Blaschke Sets for Bergman Spaces

by Boris Korenblum

ABSTRACT.: We characterize subsets S of the open unit disk $\mathbf{D}$ such that every zero sequence for a Bergman space A^p, $p > 0$, with elements in S is Blaschke.

1. Introduction.

The following definition is an extension of the notion of a Blaschke set introduced by Krzysztof Bogdan [B].

DEFINITION: We call $S \subset \mathbf{D}$ a Blaschke set for a class X of analytic functions on $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$ if

(i) whenever $0 \not\equiv f \in X$, and $\{z_n\}_n$ are the zeros of f (counting multiplicities), with $z_n \in S$, the Blaschke condition holds:

$$\sum_n (1 - |z_n|) < \infty \ ; \tag{1}$$

(ii) whenever $Z = \{z_n\}_n$ is a Blaschke sequence (i.e. (1) holds), with $z_n \in S$, there is an $f \in X$ whose zero sequence is Z.

REMARK: If X is made up of functions of bounded Nevanlinna characteric then this definition reduces to (ii). If $H^\infty \subset X$, it reduces to (i).

EXAMPLES:

1. Every subset of $\mathbf{D}$ is a Blaschke set for H^p, $0 < p < \infty$.
2. For analytic Lipschitz classes $Lip_\alpha(\mathbf{D}), \alpha > 0$, as well as for $A^\infty = \{f : f^{(n)} \in H^\infty, \forall n \geq 0\}$, Blaschke sets are characterized by

$$\int_0^{2\pi} \log \operatorname{dist}(e^{it}, S) dt > -\infty \tag{2}$$

   where dist denotes the Euclidean distance. Note that for $Lip_\alpha(\mathbf{D})$ and A^∞ the zero sequences Z are characterized by (1) and (2), with S replaced by Z.
3. The Blaschke sets S for the class $\mathcal{D}$ of analytic functions with finite Dirichlet integral are characterized by (2) (see [B]). Note that $\mathcal{D}$-zero sequences cannot be described this way because there are $f \in \mathcal{D}$ whose zeros come arbitrarily close to every point of $\partial\mathbf{D}$ (see [C] and [SS]).

1991 Mathematics Subject Classification, Primary 30C15

The purpose of this paper is to obtain a description of the Blaschke sets for Bergman spaces $A^p(p>0)$ and growth spaces $A^{-\alpha}(\alpha>0)$. Recall that A^p consists of functions f analytic on $\mathbf{D}$ such that

$$\|f\|_p^p = \int_{\mathbf{D}} |f(z)|^p \frac{dxdy}{\pi} < \infty \ ;$$

$A^{-\alpha}$ consists of analytic functions f with

$$\|f\|_{-\alpha} = \sup\{(1-|z|)^\alpha |f(z)| : z \in \mathbf{D}\} < \infty \ ;$$

We also consider the space $A^{-\infty} = \bigcup_{\alpha>0} A^{-\alpha} = \bigcup_{p>0} A^p$.

We establish the following

<u>Theorem</u>. *A set $S \subset \mathbf{D}$ is a Blaschke set for any of the spaces $A^p, A^{-\alpha}, A^{-\infty}$ if and only if (2) holds.*

To prove this theorem we first reduce condition (2) to a form involving a collection of disjoint "tents" tightly surrounding S. The sufficiency of (2) then follows from the fact that "Stolz stars" $\mathcal{S}_F$ are $A^{-\infty}$-Blaschke sets if the entropy $\hat{\kappa}(F)$ is finite (see (3) and [HKZ]). To prove the necessity of (2) we use some density concepts first introduced in [K1] and later refined in [S] and [HKZ].

<u>Acknowledgement</u>: The suthor thanks Stefan Richter and Carl Sundberg for useful discussions. Special thanks are due to Carl Sundberg for bringing K. Bogdan's work [B] to the author's attention.

2. An equivalent form of (2).

We assume that S contains a disk centered at 0 of radius $1/\sqrt{2}$.

We need some terminology.

A <u>tent</u> is an open subset h of $\mathbf{D}$ bounded by an arc $I \subset \partial\mathbf{D}$ of length less than $\pi/2$ and two straight lines through the endpoints of I forming with I an angle of $\pi/4$. The closed arc $\overline{I}$ will be called the <u>base</u> of the tent $h = h_I$. A tent h is said to <u>support</u> S if $h \cap S = \phi$ but $\overline{h} \cap \overline{S} \neq \phi$. A finite or countable collection of tents $\{h_n\}_n$ is a <u>belt</u> if h_n are pairwise disjoint and $\bigcup_n \overline{h}_n \supset \partial\mathbf{D}$. A collection of tents $\{h_n\}_n$ is an <u>S-belt</u> if h_n are pairwise disjoint, S-supporting, and $\bigcup_n \overline{h}_n \supset \partial\mathbf{D}\backslash\overline{S}$. Note that an S-belt does not have to be a belt. If S is such that $\partial\mathbf{D}\backslash\overline{S} \neq \phi$, S-belts exist: to obtain one we start at an arbitrary point $\zeta_0 \in \partial\mathbf{D}\backslash\overline{S}$ and, moving counterclockwise, consecutively find points $\zeta_1, \zeta_2 \dots$ such that the arcs between them are the bases of S-supporting tents; then we proceed similarly from ζ_0 in the opposite direction. We thus obtain a system of tents whose bases cover a component of $G = \partial\mathbf{D}\backslash\overline{S}$. Continuing this process for all the components we obtain an S-belt.

An elementary computation shows that if $h = h_I$ is a tent supporting S then

$$-|I|\log\frac{1}{|I|} - c|I| \leq \int_I \log\ \operatorname{dist}(\zeta, S)|d\zeta| \leq -|I|\log\frac{1}{|I|} + c|I|$$

where c is a numerical constant. We thus obtain

<u>Lemma 1</u>. *Let S be a subset of $\mathbf{D}$ such that $\partial\mathbf{D}\backslash\overline{S} \neq \phi$. Let $\{h_{I_n}\}_n$ be an S-belt. Then (2) holds if and only if*

(A) the set $F_0 = \overline{S} \cap \partial\mathbf{D}$ has zero Lebesgue length;

(B)

$$\sum_n \kappa(I_n) < \infty \text{ where } \kappa(I) = |I|\log\frac{2\pi e}{|I|}\ .$$

($\kappa(I)$ is called the κ length of I).

Note that (A) and (B) together are equivalent to

$$\hat{\kappa}(F) = \int_{\partial\mathbf{D}} \log\frac{2\pi}{d(\zeta, F)}|d\zeta| < \infty \tag{3}$$

where $F = F_0 \cup \Xi$ and Ξ consists of the endpoints of those bases such that $\overline{I}_n \subset G$; d denotes the angular distance.

The quantity $\hat{\kappa}(F)$ is defined for all sets $F \subset \partial\mathbf{D}$ and is called the <u>entropy</u> of F. Closed sets with finite entropy are called <u>Beurling-Carleson sets</u>.

<u>3. Sufficiency of (3)</u>.

Let $\Xi_1 \supset \Xi$ consist of <u>all</u> endpoints of the bases I_n (including those that are in F_0). Pick an increasing sequence $F_1 \subset F_2 \subset \ldots$ of finite subsets of Ξ_1 such that $\bigcup_n F_n = \Xi_1$. Then (3) implies

$$\lim_{n\to\infty} \hat{\kappa}(F_n) = \hat{\kappa}(F)\ .$$

Each F_n determines a belt whose tents are based on complementary arcs of F_n. Let H_n be the union of these tents. (Note that some of these tents are not S-supporting because they contain ponts from S). The complement $\mathbf{D}\backslash H_n = \tau_n$ is a "Stolz Star", i.e. the union of Stolz angles with vertices in F_n and apertures of $\pi/2$.

Since $\hat{\kappa}(F_n)$ are bounded, it follows that, whenever $0 \not\equiv f \in A^{-\infty}$, the Blaschke sums for those zeros of f lying in τ_n are bounded (see [HKZ], p. 118, Theorem 4.25). We have $\sum_n \tau_n \supset S$ and $\tau_1 \subset \tau_2 \subset \ldots$, which implies that the Blaschke sum for the zeros of f lying in S is finite.

4. Necessity of (3).

Suppose now that $\hat{\kappa}(F) = \infty$. Given an arbitrary fixed $p > 0$ we are going to construct a sequence $Z = \{z_n\}_n$, $z_n \in S$, such that Z is an A^p-zero sequence but $\sum(1 - |z_n|) = \infty$. In addition to the standard tools of $A^{-\infty}$-theory (density notions, premeasures, etc.) we will use some technical lemmas whose proofs are deferred to section 5.

Recall that $F = F_0 \cup \Xi$ where $F_0 = \overline{S} \cap \partial\mathbf{D}$ and Ξ is a finite or countable set lying in $G = \partial\mathbf{D} \backslash F_0$. The cluster points (if any) of Ξ are in F_0.

We consider separately two cases depending on whether $\hat{\kappa}(F_0)$ is infinite or finite.

Case 1: $\hat{\kappa}(F_0) = \infty$. By Lemma 2, s.5, there is a sequence $\{\zeta_\nu\}_1^\infty$ of distinct points in F_0 such that the corresponding arcs $\{J_\nu\}_1^\infty$ between ζ_ν and $\zeta_{\nu+1}$ are pairwise disjoint, cover $\partial\mathbf{D}$, i.e. $\bigcup_\nu \overline{J}_\nu = \partial\mathbf{D}$, and $\hat{\kappa}(\{\zeta_\nu\}) = \infty$, which is equivalent to

$$\sum_{\nu=1}^{\infty} \kappa(J_\nu) = \sum_{\nu=1}^{\infty} |J_\nu| \log \frac{2\pi e}{|J_\nu|} = \infty \ .$$

(Note that $\lim_{\nu\to\infty} \zeta_\nu = \zeta_1$). Construct a premeasure (see [K1], [K2], [HKZ]) $d\mu = p|d\zeta| - d\sigma$ whose positive part has the density

$$p(\zeta) = \log \frac{2\pi}{d(\zeta, \{\zeta_\nu, \zeta_{\nu+1}\})}, \zeta \in J_\nu,\ \nu \geq 1 \ ,$$

and the negative singular part $-d\sigma$ puts on every point ζ_ν a Dirac mass equal to $-\kappa(J_\nu)$. Although both positive and negative parts are infinite on $\partial\mathbf{D}$, $d\mu$ is κ-bounded above, which means that there is a constant $c > 0$ such that for all arcs $I \subset \partial\mathbf{D}$

$$\mu(I) \leq c|I| \log \frac{2\pi e}{|I|} \ .$$

This enables us to consider a zero-free analytic function

$$f_\varepsilon(z) = \exp\{\varepsilon \int_{\partial\mathbf{D}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta)\}$$

which is in A^p (or $A^{-\alpha}$) provided that $\varepsilon > 0$ is sufficiently small, and p (or α) are arbitary but fixed positive numbers.

Now we use Lemma 3, s.5, to reduce all the singular masses at ζ_ν by a factor 1/2 and compensate for that by extra zeros of high multiplicity at $z_\nu \in S$. We can ensure that the resulting function Φ is in A^p. The zeros z_ν of ϕ (counting multiplicities) form a non-Blaschke sequence of points from S (see Lemma 3 for details).

Case 2: $\hat{\kappa}(F_0) < \infty$. Then we must have $\hat{\kappa}(\Xi) = \infty$. Recall that Ξ includes all the endpoint of the base arcs of the S-belt that are in $G = \partial\mathbf{D}\backslash\overline{S}$. Let $\{J_\nu\}_\nu$ be the sequence of these arcs arranged by decreasing lengths. Then $\hat{\kappa}(F_0) < \infty$ together with $\hat{\kappa}(\Xi) = \infty$ yield

$$\sum_{\nu=1}^{\infty} \kappa(J_\nu) = \sum_{\nu=1}^{\infty} |J_\nu| \log \frac{2\pi e}{|I_\nu|} = \infty \ .$$

It is always possible to find a decreasing sequence $1 > \lambda_1 > \lambda_2 > \ldots \to 0$ such that

$$\sum_{\nu=1}^{\infty} \lambda_\nu \kappa(J_\nu) = \infty \ .$$

Every $\overline{J}_\nu$ is the base of a tent h_{J_ν} that supports S; therefore there is at least one common point, say w_ν, in $\overline{S}$, $\overline{h}_{J_\nu}$ and $\mathbf{D}$. Take every w_ν and repeat it $[\frac{\lambda_\nu \kappa(J_\nu)}{1-|w_\nu|}]$ times. Let the resulting sequence be $Z = \{z_k\}_k$.

Claim: Z is a zero sequence for every A^p, $p > 0$. To prove the claim we employ the notion of upper asymptotic κ-density of a sequence in $\mathbf{D}$. There are several equivalent definitions of this density. We will use the definition based on radial stars (see [HKZ]):

For an arbitrary finite set $M \subset \partial\mathbf{D}$ let r_M denote the union of radii from 0 to points in M. If $A = \{a_k\}_k$ is any sequence of points in $\mathbf{D}$, we form the partial Blaschke sum for A and $r_{\mathcal{M}}$:

$$B(A, r_M) = \sum_{\kappa} \{1 - |a_k| : a_k \in r_M\} \ ,$$

and define

$$D^+(A) = \limsup_{\hat{\kappa}(M)\to\infty} \frac{B(A, r_M)}{\hat{\kappa}(M)} \tag{4}$$

where lim sup is taken over all finite $M \subset \partial\mathbf{D}$.

The following result, although short of a full characterization of A^p-zero sets, is sharp enough for our purposes.

Proposition. *(See [HKZ], p.130, Theorem 4.37). Let $A = \{a_\kappa\}_k$ be a sequence of points in $\mathbf{D}$ and $\mathbf{D}^+(A)$ be the upper asymptotic κ-density of A. If $D^+(A) < \frac{1}{p}$ then A is an A^p-zero sequence. If $D^+(A) > \frac{1}{p}$ then A is not an A^p-zero sequence.*

Remark: This is a sharper version, due to Kristian Seip [S], of a simiilar but weaker result from [K1].

Now we can prove the claim by showing that $D^+(Z) = 0$. Let $Q = \{q_\nu = \frac{w_\nu}{|w_\nu|}\}_\nu$. Every arc J_ν contains exactly one point from Q, namely q_ν. Obviously, for computing

$D^+(Z)$ we can employ only those M that are finite subsets of Q. For such M we have

$$B(Z, r_M) \le \sum_\nu \{\lambda_\nu \kappa(J_\nu) : q_\nu \in M\}$$

and

$$\hat{\kappa}(M) \ge \sum_\nu \{\kappa(J_\nu) : q_\nu \in M\}$$

(see Lemma 4, s.5). Therefore

$$\frac{B(Z, r_M)}{\hat{\kappa}(M)} \le \sum_\nu \{\lambda_\nu \kappa(J_\nu) : q_\nu \in M\} / \sum_\nu \{\kappa(J_\nu) : q_\nu \in M\} \ .$$

It is convenient to use the following notations:

$$K(M) = \sum_\nu \{\kappa(J_\nu) : q_\nu \in M\} \ ,$$

$$K_\lambda(M) = \sum_\nu \{\lambda_\nu \kappa(J_\nu) : q_\nu \in M\} \ .$$

Let $\{M_n\}_n$ be a sequence of subsets of Q such that $\hat{\kappa}(M_n) \to \infty$. Then we have

$$\frac{B(Z, r_{M_n})}{\hat{\kappa}(M_n)} \le \frac{K_\lambda(M_n)}{K(M_n)} \ . \tag{5}$$

Suppose that $K(M_n) = \mathcal{O}(1)(n \to \infty)$. Then obviously the left-hand side of (5) tends to 0. Also, if $K(M_n) \to \infty$, then the right-hand (and, with it, the left-hand) side of (5) tends to 0 because $\lambda_\nu \downarrow 0$. Therefore every sequence $\{M_n\}$, $M_n \subset Q$, $\hat{\kappa}(M_n) \to \infty$, contains a subsequence $\{M_{n_k}\} = \{M'_k\}$, $n_1 < n_2 \ldots$, such that

$$\lim_{k \to 0} \frac{B(Z, r_{M'_k})}{\hat{\kappa}(M'_k)} = 0 \ .$$

which implies $D^+(Z) = 0$. Thus we have obtained a non-Blaschke A^p-zero sequence $\{z_k\}$ whose elements are in $\overline{S}$. Using how Lemma 5, s.5, we can replace z_k by nearby points $\tilde{z}_k$ from S so that the new sequence $\{\tilde{z}_k\}_k$ is still an A^p-zero sequence and non-Blaschke. This completes the proof of the Theorem.

5. Technical Lemma.

We give below the statement of the technical lemmas we used in proving the Theorem, together with a brief outline of their proofs.

DEFINITION: A sequence $\{\zeta_n\}_1^\infty$ of distinct points in $\partial\mathbf{D}$ is called **T**-monotone if the open arcs I_n between ζ_n and ζ_{n+1} are pairwise disjoint and $\bigcup_n \overline{I} = \partial\mathbf{D}$. Note that it follows from this definition that $\lim_{n\to\infty} \zeta_\infty = \zeta_1$.

LEMMA 2. *Every closed set $F \subset \partial\mathbf{D}$ of infinite entropy contains a $\mathbf{T}$-monotone sequence $\{\zeta_n\}_n \subset F$ of infinite entropy.*

PROOF: We have

$$\hat{\kappa}(F) = \int_{\partial\mathbf{D}} \log \frac{2\pi}{d(\zeta, F)} |d\zeta| = \infty \ .$$

(d denotes the angular distance). By the Heine-Borel lemma there is a point $\zeta_0 \in F$ such that every open arc J containing ζ_0 has the property

$$\int_J \log \frac{2\pi}{d(\zeta, F)} |d\zeta| = \infty \ .$$

Now we can find a nested system of open arcs such that $J_\kappa \supset \overline{J}_{n+1}$, $\bigcap_n I_n = \{\zeta_0\}$, and a finite set $M_k \subset (J_n \backslash \overline{J}_{n+1}) \cap F$ such that

$$\int_{J_n \backslash \overline{J}_{n+1}} \log \frac{2\pi}{d(\zeta, M_n)} |d\zeta| \geq 1, \ n \geq 1 \ .$$

Taking the union $E = \bigcup_n M_n$ (or a suitable subset of E) and rearranging it in a sequence will prove the Lemma.

LEMMA 3. *Let $f \in A^p (p > 0)$ have an "atomic singularity" at $z = 1$, i.e.*

$$\limsup_{r \to 1^-} (1 - r) \log |f(r)| = -2m < 0 \ .$$

If $m_1 < m$ then

(i) $F(z) = e^{m_1 \frac{1+z}{1-z}} f(z)$ is in A^p;

(ii) whenever $0 \neq \alpha_n \in \mathbf{D}$ and $\lim_{n \to \infty} \alpha_n = 1$, the function

$$f_{\alpha_n}(z) = \left(\frac{\alpha_n - z}{1 - \overline{\alpha}_n z} \cdot \frac{|\alpha_n|}{\alpha_n}\right)^{N_n} F(z), \ \text{where } N_n = \left[\frac{m_1}{1 - |\alpha_n|}\right] \ ,$$

tends to f in the metric of A^p.

PROOF: (i) For any $r \in (0, \infty)$ the equation

$$\frac{1 - |z|^2}{|1 - z|^2} = r$$

defines a circle C_r internally tangent to $\partial\mathbf{D}$ at the point 1. Such circles are called orocycles. If f is in A^p and has atomic singularity m at 1, then the function

$$g(z) = e^{m \frac{1+z}{1-z}} f(z)$$

may not be in A^p; however, the integral

$$L(r) = \frac{1}{2\pi} \int_{C_r} |1-\zeta|^2 |g(\zeta)|^p |d\zeta|$$

is finite and decreasing on $(0, r)$, and

$$\int_{\mathbf{D}} |f(z)|^p \frac{dxdy}{\pi} = \int_0^\infty e^{-mr} L(r) dr < \infty \ .$$

This implies

$$\int_{\mathbf{D}} |F(z)|^p \frac{dxdy}{\pi} = \int_0^\infty e^{-(m-m_1)r} L(r) dr < \infty \ .$$

(ii) then follows by the dominated convergence theorem

LEMMA 4. *If $I \subset \partial\mathbf{D}$ is an arc, M is an arbitrary subset of $\partial\mathbf{D}$, and if at least one point from M is in $\overline{I}$, then*

$$\int_I \log \frac{2\pi}{d(\zeta, M)} |d\zeta| \geq \kappa(I) = |I| \log \frac{2\pi e}{|I|} \ .$$

PROOF: The minimum of the integral on the left for a given arc I is attained when M is a one-point set, and this point is one of the endpoints of I. A direct computation yields the required result.

LEMMA 5. *Let $f \in A^p$ have a zero at some point $a \in \mathbf{D}$. For arbitrary $\alpha \in \mathbf{D}$ define*

$$f_\alpha(z) = \frac{B_\alpha(z)}{B_a(z)} f(z)$$

where B is a Blaschke factor:

$$B_\alpha(z) = \frac{z-\alpha}{1-\overline{\alpha}z}, \ B_a(z) = \frac{z-a}{1-\overline{a}z} \ .$$

Then f_α tends to f in the metric of A^p as $\alpha \to a$.

The proof is immediate and left to the reader.

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