Personal Research Interests and Activity

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Table of Contents

1 General

The range of my strictly mathematical research interests falls within the territory spanned by number theory and algebraic geometry, and I have since 1987 acquired an interest in the meta-mathematical topic of the design of "mathematics in the international universe of networked computing".

Most of my visible past strictly mathematical investigations have involved modular forms, theta functions, abelian varieties, the geometry of Hilbert modular surfaces, and "reduction" of Schwartz-Bruhat functions.

2 Mathematics on the Network

In the arena of "mathematics and the network" great hope has been present since, say, 1993 when CERN's Tim Berners-Lee's free World Wide Web (WWW) emerged into the consciousness of the mathematical community at large.

From the University of Minnesota Paul Lindner offered "gopher", an alternative approach with browsing free to everyone and serving free to universities, and with by 1993 the handling of **arbitrary** "content-types". This alternative had equal potential utility to serious needs of academic mathematics, science, and engineering.

In a sense not much has happened since that time, while a great deal more[1] is explicitly desired by the mathematical community. It is my opinion that much more will happen if the design of "mathematics" on the web is regarded as a question of the construction of an abstract meta-mathematical entity. It may not be too late. It does need the attention of mathematicians who understand the universe of networked computing.

Beginning in late 1987 I became interested in optimizing the use of desktop computer processing for the authoring and reading of mathematical documents. Many of my ideas about Mathematical Typewriter Emulation (MTE)[2] were formed at that time. Those ideas are evolving inasmuch as they are still somewhat relevant to current issues concerning math on the web.

The question is how to design the whole structure, as an abstract entity, of mathematical information in the universe of networked computers so as to enable that universe:

- to contain all information that in 1970 one expected to be able to find available in printed paper form in library buildings.
- to admit sufficiently complex searching that a mathematician may freely locate sources in that universe of current information on a narrowly defined mathematical question beginning from little information and without reliance on human networks.

This has led to what I call "GELLMU"[3], a name which, as I use it, refers to:

1. a cross-platform free[4] computer software project assembled with interchangeable components.
2. an extensible LaTeX-like markup language with provision for mathematical authors serving as *A Bridge from LaTeX to XML*[5] that is suitable for *single source authoring toward multiple presentation formats* (SSATMF).

3 Reduction of Schwartz Bruhat Functions

Despite all of my work with computing and "mathematics on the web", some of which includes

[1] URI: http://www.albany.edu/~hammond/gellmu/webmath.html
[2] URI: mte.html
[3] URI: igl.html
[4] URI: http://www.gnu.org/
[5] URI: http://www.tug.org/TUGboat/Articles/tb22-3/tb72hammond.pdf

very substantial voluntary service to my Department in supervising its *IX network since 1992 and serving as both editor and manager of a substantial portion of its web service, I do have a current mathematical focus, which since 1977, has been the topic of "product formulas", a phrase that in my context is implied jargon referring to identities that involve one of a certain fuzzy family of infinite products indexed by *primes*, often called "Euler products".

A simple, but important, special case of such products is an essential invariant of the geometric object that is our contemporary understanding of the *solution* of a finite number of polynomial equations in finitely many variables. And, of course, there are important "modular forms" and important objects in "representation theory" that are such products.

Reduction of Schwartz-Bruhat functions is a topic handled explicitly in Donald Roby's 1980 Albany thesis, under my direction, that arises naturally as an extension of J.-I. Igusa's extension of the principal result of André Weil (1906-1998) in his *Acta* paper (1964). These results apply to the context of *adelic analysis*, a (the?) context from which product formulas arise and which *should* be understood today as part of "basic analysis"[6], though that perception has yet to receive a proper dissemination in graduate-level mathematical education inside many institutions in the United States.

Weil, as an easy sideline corollary, reached the only truly conceptual proof (at least that I know) of (Hilbert's product formula version of) Gauss's famous law of quadratic reciprocity.

In the early years after the particular Weil results (his first *Acta* paper published in 1964) became known, there was a substantial "buzz", especially around Harvard, about the possible relation of adelic Haar measure to the study of the "L-function" of an elliptic curve. This view led eventually to the "Langlands Program".[7]

The approach to the modular curve theorem through adelic-analysis, in contrast with the recent successful approach through equivariant arithmetic geometry, so far has not born fruit. I still see the absence of such a proof of this basic fact about elliptic curves as a hole in our knowledge of "basic analysis" without knowing what to do about it. The precise Archimdean dimensions of this hole were measured by Weil and published in *Math. Ann.* in 1967. The subject of elliptic curves is rich enough to admit more than one approach.

It is, even today after the work of Wiles, not completely clear that there is no use to be found in the circle of ideas surrounding "reduction of Schwartz-Bruhat functions" in the subject of elliptic curves. This question may be regarded as a question about the transport of the *reduction* idea to non-abelian contexts with arithmetic dimension 2. In fact, a useful connection, if any, of this idea with Artin reciprocity (see below), a context of arithmetic dimension 1, has yet to surface, so far as I know. It may take time.

A proper understanding of basic analysis here will certainly take time. I suspect that in

[6] URI: anabasic.html

[7] It even led eventually by the time of the Antwerp Meeting on Modular Forms in 1972 to an inadvertently inappropriate use of the term *Weil curve* (not, I believe, originating with Weil) for an elliptic curve that arises from a modular form. The name "Shimura curve" would have been better except that at the time it had another meaning; today one says "modular curve" for such a curve, which is understood now as an elliptic curve that is, up to isogeny, in the image of a *map* — the "Shimura map" — first constructed by Goro Shimura that might have been given more highlighting in his 1971 book. It was reported in 1999 that Breuil, Conrad, Diamond, and Taylor had proved the modular curve conjecture using an argument along the lines outlined by Wiles in 1993 for the case of a "semi-stable" elliptic curve.

both cases (reciprocity and elliptic) one wants to look far beyond the Heisenberg group and its Schrödinger normalizer. One wants to be alert for interesting operators of finite order, especially order 6 in the elliptic case. (Fourier transform is an operator of order 4 that is almost certainly relevant; of course, it *is* relevant for theta *nullwerte*.)

In the general theory of product formulas, one wants to understand the entire class of identities that ultimately may be regarded as consequences (perhaps, for some cases just in a mirror-like fashion) of two things:

1. the fact that Haar measure on the adeles is **restricted product measure**.
2. in-the-case evaluation of integrals, i.e., adelic and everywhere local **evaluation of integrals** (whether or not L^1, $\{\text{i.e.}\}^2$, "absolutely convergent") that are derived from objects existing over the rational field (or, as appropriate for the context, a "global field" of characteristic zero).

This is a fundamental task in basic analysis.

I expect that André Weil, who appears to me to have been careful about venturing "conjectures", as opposed to raising questions, especially perhaps after having perceived a loss with "intermediate Jacobians" in the 1960's, imagined this as a further task when he posed as a challenge in his *Acta* paper the explicit task of bringing Artin reciprocity under this banner inasmuch as this is the most obvious more general framework to consider when pondering his treatment of the Hilbert product formula.

We do not have proper follow-up on Roby's work.

4 Everyone Seminar, October 1993

The official notes[8] on my October 22, 1993 Albany *Everyone Seminar* presentation offer a beginning introduction to the idea that Fermat's Last Theorem is a consequence of knowing as much about plane curves defined by cubic equations with *rational coefficients* as first year undergraduates have traditionally been expected to know about plane curves defined by quadratic equations with real coefficients.

5 Special Theta Relations

The notes of my 1988 New York City Number Theory Seminar talk have have finally appeared: "Special Theta Relations", Number Theory: New York Seminar 1991-1995, D. V. Chudnovsky et al., eds., Springer Verlag, 1996, pp. 195-199.

As far as I know, Patrick McNally's 1995 doctoral dissertation on the subject has not appeared. For me its chief thrust is that in the context of abelian surfaces with real quadratic endomorphisms the classically-described theta relations obtained as a corollary of Mumford's *Tata*

[8] URI: oct93.html

approach to Riemann's relations (my New York talk) have the same zero loci as the complete set of relations constructed somewhat less classically (in the *newer* more general Mumford setting) by Zarhin.

Of course, the whole subject of theta functions has deep ties to the subject of "Basic Analysis", properly understood.

6 Plans Regarding "Reduction"

Unless Roby, unbeknownst to me, has published his thesis, I plan an expansion of some of the material treated there and never published beyond archiving at University Microfilms. I have been stalling this writing project since about 1990 while seeking a **markup language worthy of the effort**, leading to my development of GELLMU[9], which, as of 2004, is adequate to this task.

7 Weil's Real Metaplectic Group

Some day there will be available here an item from 1976[10], never published, dealing with the straightforward but tedious calculation of the 2-cocycle giving the Shale-Mackey-Weil group as an extension of the group of centrally trivial automorphisms of the real Heisenberg group by the one dimensional compact real torus that provided my motivation for one of the questions treated in Roby's thesis. The p-adic case was treated in Roby's thesis. This also has been waiting for GELLMU.

8 A Certain Class of Primes

During 1991-92 I went searching for the second member[11] of a class of primes that ought to be infinite. This is the class of odd primes p for which the smallest positive primitive root c mod p fails also to be primitive mod p^2. Such failure is equivalent to the condition

$$c^{p-1} \equiv 1 \pmod{p^2} \quad .$$

In the common case when a number is primitive mod p^2, p an odd prime, it is necessarily primitive modulo every power of p, and, therefore, it is a topological generator of the multiplicative group of p-adic units. The smallest member of the class is the prime 40487. In April 2001, Professor Stephen Glasby of the University of Central Washington wrote me that he had found another example, which is the prime 6692367337, and that these two primes are the only examples smaller than 10^{10}.

[9]URI: http://www.albany.edu/~hammond/gellmu/
[10]URI: rmpgp.tex
[11]URI: cpe-nth

9 Planned Course Notes on Theta Functions

Also I plan to make available here my notes from the course about theta functions that I presented in the Spring of 1995. This will require converting sketchy outlines to markup; it has very low priority right now. Like any "busy-work" task on my list it is always at risk of yielding to something more worthwhile.

10 Syracuse/3n+1 Doodles

In 1992 I wrote some `"gp"` code (as in "PARI/gp" from Henri Cohen et al., see, e.g., the locally archived information[12] regarding PARI) for toying with the problem known variously as "Syracuse", "$3N + 1$", ... Since I used it for only a month or two without drawing any firm conclusions and it has lain dormant since that time, I am making my package of `"gp"` code[13], which contains about 55 functions in 1200 lines of code, believed to be debugged, available with a "doc"[14] giving English function definitions but otherwise without explanation. In this I think that I may have seen a machine that makes infinite eventually periodic sequences of natural numbers, and another such machine, of course, is a quadratic Hilbert modular cusp, something subordinate to a real quadratic number field. It's just a curiosity.

An online reference for the Syracuse, "$3N + 1$", . . . problem is an article by Jeff Lagarias that originally appeared in the MAA *Monthly*:

`http://www.cecm.sfu.ca/organics/papers/lagarias/paper/html/paper.html` .

11 Command Line Utilities

Along the way I became interested in writing certain kinds of software[15], all of which had at some point been relevant to mathematical authoring. For example, `"fwid"` is relevant both to MTE and to clean `text/plain` or `text/ansi` rendering of GELLMU documents, `"conv"` is relevant both to efficient printer setting of MTE as well as to articulation between authoring platform base character sets, i.e., ASCII vs. EBCDIC, and the experience of having once needed to furnish myself with `"xcho"` led to my philosophy that *every non-word character in a platform base character set needs a symbolic name in* ***any*** *sane single-source authoring language on that platform.*

[12]URI: http://math.albany.edu:8010/g/Math/MathComp/pari/view.html
[13]URI: n32p
[14]URI: n32p-doc
[15]URI: sftw.html

12 About this document

This is an example of a GELLMU[16] document. Various forms of it are available: the GELLMU source[17], the syntactic translation[18] to SGML, the subsequent translation[19] to XML from which all end formattings are derived, the HTML formatting[20], the LaTeX formatting[21], a DVI file for "letter" paper[22], a PDF file[23], and an XHTML+MathML formatting[24].

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[16] URI: igl.html
[17] URI: research/rsch.glm
[18] URI: research/rsch.sgml
[19] URI: research/rsch.xml
[20] URI: research/rsch.html
[21] URI: research/rsch.ltx
[22] URI: research/rsch.dvi
[23] URI: research/rsch.pdf
[24] URI: http://math.albany.edu:8010/math/pers/hammond/research/rsch.xhtml