

Review of Curve and Surface Integrals

for Calculus III (Math 214)

Definitions

Curve integrals

$C: \mathbf{r}(t), a \leq t \leq b$

(a) scalar function f :

$$\int_C f \, ds = \int_a^b f(\mathbf{r}(t)) \|\mathbf{r}'(t)\| \, dt$$

(b) vector function F :

$$\int_C F \cdot d\mathbf{r} = \int_a^b F(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt$$

Note: $d\mathbf{r} = \mathbf{r}'(t) \, dt = \mathbf{T} \, ds$; $ds = \|d\mathbf{r}\| = \|\mathbf{r}'(t)\| \, dt$

Surface integrals

$S: \mathbf{W}(u, v), (u, v) \in E$ (E a planar region)

(a) scalar function f :

$$\iint_S f \, d\sigma = \iint_E f(\mathbf{W}(u, v)) \|\mathbf{W}_u \times \mathbf{W}_v\| \, dudv$$

(b) vector function F :

$$\iint_S F \cdot d\mathbf{W} = \iint_E F(\mathbf{W}(u, v)) \cdot (\mathbf{W}_u \times \mathbf{W}_v) \, dudv$$

Note: $d\mathbf{W} = (\mathbf{W}_u \times \mathbf{W}_v) \, dudv = \mathbf{N} \, d\sigma$; $d\sigma = \|d\mathbf{W}\| = \|\mathbf{W}_u \times \mathbf{W}_v\| \, dudv$
where

$$W_u = \frac{\partial \mathbf{W}}{\partial u}, \quad W_v = \frac{\partial \mathbf{W}}{\partial v}$$

Special case 1: graph of a function of two variables

$$S: z = f(x, y), (u, v) = (x, y), \mathbf{W}(x, y) = (x, y, f(x, y)), d\mathbf{W} = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right) dx dy$$

Special case 2: a sphere parameterized with spherical coordinates

$$\mathbf{W}(u, v) = a(\cos u \sin v, \sin u \sin v, \cos v) \quad d\mathbf{W} = -a^2 \sin v (\cos u \sin v, \sin u \sin v, \cos v) \, du \, dv$$

For both curves and surfaces a parameterization determines an orientation. The orientation of a surface in \mathbf{R}^3 is determined by the two-fold choice of a unit normal to that surface, and the boundary of an oriented surface is given the orientation that is related to the chosen unit normal in a right-handed coordinate system by the “right-hand rule”.

The Fundamental Theorem of Calculus

Theorems of the form $\int_{\partial G} \omega = \int_G d\omega$

dim G	ω	$d\omega$	left side	right side	Remarks
1	f	f'	$f(b) - f(a)$	$\int_I f'(t) dt$	Fund. Thm. of Calculus I interval in \mathbf{R} from a to b
1	f	∇f	$f(B) - f(A)$	$\int_C \text{grad} f \cdot d\mathbf{r}$	C path in \mathbf{R}^n from A to B
2	$F \cdot d\mathbf{r}$	$(\nabla \wedge F) dA$	$\int_{\partial R} F \cdot d\mathbf{r}$	$\iint_R \text{curl} F dA$	Green's Thm. R region in \mathbf{R}^2 (∂R anti-clockwise in right-hand coord. system)
2	$F \cdot d\mathbf{r}$	$(\nabla \times F) \cdot d\mathbf{W}$	$\int_{\partial S} F \cdot d\mathbf{r}$	$\iint_S \text{curl} F \cdot d\mathbf{W}$	Classical Stokes' Thm. S surface in \mathbf{R}^3 (with r/l hand rule)
3	$F \cdot d\mathbf{W}$	$(\nabla \cdot F) dV$	$\iiint_{\partial D} F \cdot d\mathbf{W}$	$\iiint_D \text{div} F dV$	The Divergence Thm. D domain in \mathbf{R}^3 (∂D with outer normal)