Math 520B

Written Assignment No. 4

due Friday, November 18, 2005

Directions. Although you may refer to books for definitions and standard theorems, searching for solutions to these written exercises is **not** permitted. You may not seek help from others.

Bear in mind that rings are always assumed to have a multiplicative identity, and a homomorphism of rings is always assumed to carry the multiplicative identity of its domain to that of its target. Recall that if T is a ring, the term T-algebra indicates, by definition, a pair $\langle R, \rho \rangle$ where R is a ring and $\rho : T \to R$ is a ring homomorphism.

1. Let F be a field, and let V and W be finite-dimensional vector spaces over F. Recall that the ring of endomorphisms of an F-vector space is an F-algebra.

   (a) Explain why for $f \in \mathrm{End}_F(V)$ and $g \in \mathrm{End}_F(W)$ there is a unique $h = f \otimes g \in \mathrm{End}_F(V \otimes_F W)$ for which
   $$h(x \otimes y) \;=\; f(x) \otimes g(y) \quad .$$

   (b) Show that the map
   $$\mathrm{End}_F(V) \times \mathrm{End}_F(W) \longrightarrow \mathrm{End}_F(V \otimes_F W)$$
   given by $(f, g) \mapsto f \otimes g$ is F-bilinear.

   (c) Prove that the bilinear map in the previous part provides an isomorphism
   $$\mathrm{End}_F(V) \otimes_F \mathrm{End}_F(W) \longrightarrow \mathrm{End}_F(V \otimes_F W) \quad .$$

2. Let F be a field, V an n-dimensional vector space over F, $\otimes^p V$ the vector space
   $$\otimes^p V \;=\; \underbrace{V \otimes_F V \otimes_F \ldots \otimes_F V}_{p \text{ times}}$$
   with the convention $\otimes^0 V = F$, and $T(V)$ the direct sum
   $$T(V) \;=\; \bigoplus_{p \geq 0} \otimes^p V \quad .$$
   Endow $T(V)$ with the structure of a non-commutative F-algebra as follows:

   (a) Define canonical bilinear maps
   $$\otimes^p V \times \otimes^q V \longrightarrow \otimes^{p+q} V$$

   (b) Use the bilinear maps of the previous item with standard facts about direct sums to define multiplication
   $$T(V) \times T(V) \longrightarrow T(V) \quad .$$

3. With F, V, and $T(V)$ as in the previous exercise, do the following:

   (a) Prove that if V is 1-dimensional over F, then $T(V)$ is isomorphic to the polynomial ring $F[t]$.

   (b) State and prove a universal (initial) mapping property for the *tensor algebra* $T(V)$.

4. Let A be a commutative ring and I, J ideals in R. Prove that
   $$A/I \otimes_A A/J \;\cong A/(I + J) \quad .$$

5. Let F be a field, $i : F[t] \to F[x, y]$ the unique F-algebra homomorphism for which $i(t) = x$ and $j : F[t] \to F[x, y]$ the unique F-algebra homomorphism for which $j(t) = y$.

   Let V be the F-vector space having basis $\{X, Y\}$ which may be canonically identified with the subspace $\otimes^1 V \subset T(V)$. Let $i' : F[t] \to T(V)$ be the unique F-algebra homomorphism for which $i'(t) = X$ and $j' : F[t] \to T(V)$ be the unique F-algebra homomorphism for which $j'(t) = Y$.

   (a) Show that $(F[x, y], i, j)$ is universal-initial among triples (X, f, g) where X is a centered F-algebra, and f, g are F-algebra homomorphisms satisfying $f(p)g(q) = g(q)f(p)$ for all $p, q \in F[t]$.

   (b) Show that $(T(V), i', j')$ is universal-initial among triples (X, f, g) where X is a centered F-algebra, and f, g are any F-algebra homomorphisms.