Math 520B

Written Assignment No. 3

due Monday, October 31, 2005

Directions. Although you may refer to books for definitions and standard theorems, searching for solutions to these written exercises is **not** permitted. You may not seek help from others.

Bear in mind that rings are always assumed to have a multiplicative identity, and a homomorphism of rings is always assumed to carry the multiplicative identity of its domain to that of its target. Recall that if T is a ring, the term T-algebra indicates, by definition, a pair $\langle R, \rho \rangle$ where R is a ring and $\rho : T \to R$ is a ring homomorphism.

1. If F is a field and $n > 0$ an integer, it has been explained in this course that an $F[t]$-module structure on F^n extending its usual F-module structure amounts to the same thing as an $n \times n$ matrix in F. Show that the $F[t]$-modules corresponding to two given $n \times n$ matrices A and B are isomorphic if and only if A and B are similar over F.

2. For a prime $p > 1$ it is a fact that the sequence of coefficients in the p-adic expansion of a p-adic integer that is rational must eventually repeat.

   (a) Find the 5-adic expansion of the rational number $2/3$.

   (b) Find the expansion of $2/3$ as a real "decimal" computed in base 5.

3. If T is a ring, employ a suitable universal mapping property to state what is meant by an abstract product of a non-empty collection of T-algebras. Then prove that such always exist.

4. Employ a suitable universal mapping property to state what is meant by a direct limit for a non-empty collection of rings indexed by a directed set with a suitable collection of connecting ring homomorphisms. Use this definition in carrying out the following task.

   Let A be a domain, K its field of fractions, and P a prime ideal in A. Let A_P denote the subring of K consisting of those elements of K representable as a/b where $a, b \in A$ with $b \notin P$, and for $f \in A$ let A_f denote the subring of K consisting of those elements of K representable as a/f^n for some integer $n > 0$. Note that when g is a multiple of f, there is an inclusion homomorphism from A_f to A_g. Show that A_P is a direct limit of $\{ A_f \mid f \notin P \}$.

5. Let F be a field, V, W finite-dimensional vector spaces over F, and $\check{V}$ the dual of V. Prove that
$$T \ = \ \mathrm{Hom}_F(\check{V}, W)$$
together with the bilinear map $\tau : V \times W \to T$ defined by
$$(v, w) \longmapsto [\phi \mapsto \phi(v) \cdot w]$$
is a tensor product of V and W over F.