Math 520A Written Assignment No. 1

due Wednesday, February 14, 2007

Directions. This assignment should be typeset. If you make use of a reference other than class notes, you must properly cite that use. You may not seek help from others.

Notation: Let F be a field. The following notations will be used.

F^*	the multiplicative group of F
$\mathrm{Mat}_n(F)$	the ring of all $n \times n$ matrices in F
$\mathrm{GL}_n(F)$	the multiplicative group $(\mathrm{Mat}_n(F))^*$
det	the homomorphism $\mathrm{GL}_n(F) \to F^*$ given by taking the *determinant* of a matrix
$\mathrm{SL}_n(F)$	the kernel of the homomorphism det
ν_n	the homomorphism $F^* \to \mathrm{GL}_n(F)$ given by $a \mapsto a \cdot 1_n$, (1_n the identity)
$\mathrm{PGL}_n(F)$	the quotient group $\mathrm{GL}_n(F)/\mathrm{Im}(\nu_n)$

1. For each infinite field F and each integer $n \geq 2$ provide an example of an infinite subgroup of the group $\mathrm{GL}_n(F)$ of invertible $n \times n$ matrices in F that both contains a finite subgroup isomorphic to the group of permutations of the n coordinate axes of F^n and is not a normal subgroup of $\mathrm{GL}_n(F)$.

2. Let R be a commutative ring and I an ideal in R. One says that two matrices A and B in $\mathrm{Mat}_n(R)$ are *congruent* modulo I if the difference matrix $A - B$ has entries in I. Show that the set of matrices congruent to 0 modulo I is a two-sided ideal J_n in $\mathrm{Mat}_n(R)$, and describe the quotient ring $\mathrm{Mat}_n(R)/J_n$.

3. Determine the number of isomorphism classes among commutative rings (having 4 elements) of the form

$$\mathbf{F}_2[t]/(t^2 + at + b)\mathbf{F}_2[t]$$

   where $\mathbf{F}_2$ is the field with 2 elements, $\mathbf{F}_2[t]$ the ring of polynomials with coefficients in $\mathbf{F}_2$, and $a, b \in \mathbf{F}_2$.

4. Let $\mathbf{F}_3$ denote the field of 3 elements. Observe that the order of the group $\mathrm{GL}_2(\mathbf{F}_3)$ is 48 and that the groups $\mathrm{SL}_2(\mathbf{F}_3)$ and $\mathrm{PGL}_2(\mathbf{F}_3)$ both have order 24. The group S_4 of all permutations of a set of 4 elements also has order 24. Determine which, if any, of these three groups of order 24 are isomorphic.

5. Let F be a field, and let V be a finite-dimensional vector space over F. V^* will denote the dual space of V. The Heisenberg group $\mathrm{Hs}(V)$ is the set $V \times V^* \times F$ with group law given by

$$(v_1, f_1, t_1) * (v_2, f_2, t_2) \quad = \quad (v_1 + v_2, f_1 + f_2, t_1 + t_2 + f_2(v_1)) \quad .$$

   (a) Show that the center[1] C of $\mathrm{Hs}(V)$ is the set

$$\{(0, 0, t) \in \mathrm{Hs}(V) \mid t \in F\} \quad .$$

   (b) Let H denote the set

$$\{(0, f, 0) \in \mathrm{Hs}(V) \mid f \in V^*\} \quad .$$

   Show that H is a subgroup of $\mathrm{Hs}(V)$ that is isomorphic to the additive group of V^*.

   (c) Let N denote the set

$$\{(v, 0, t) \in \mathrm{Hs}(V) \mid v \in V,\ t \in F\}$$

   Show that N is a normal subgroup of $\mathrm{Hs}(V)$.

   (d) Show that the quotient $\mathrm{Hs}(V)/N$ is isomorphic to V^*.

   (e) Since N is normal in $\mathrm{Hs}(V)$, the subgroup H conjugates N to itself, i.e., one has $hnh^{-1} \in N$ for all $n \in N$ and all $h \in H$, and this provides an action of H on N. Describe the action of (the additive group of) V^* on N that corresponds via the isomorphism of H with V^* to this action of H on N.

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[1]Definition. The *center* of a group is the subset of the group consisting of those elements that commute with every element of the group.