Math 520A Written Assignment No. 1

due Monday, February 14, 2005

Directions. It is intended that you work these as exercises. Although you may refer to books for definitions and standard theorems, searching for solutions to these written exercises either in books or in online references should not be required and is undesirable. If you make use of a reference other than class notes, you must properly cite that use.

You may not seek help from others.

Notation: Let F be a field. The following notations will be used.

F^*	the multiplicative group of F
$\mathrm{GL}_n(F)$	the group of all $n \times n$ invertible matrices in F
det	the homomorphism $\mathrm{GL}_n(F) \to F^*$ given by taking the *determinant* of a matrix
$\mathrm{SL}_n(F)$	the kernel of the homomorphism *det*
ν_n	the homomorphism $F^* \to \mathrm{GL}_n(F)$ given by $a \mapsto a \cdot 1_n$, (1_n the identity)
$\mathrm{PGL}_n(F)$	the quotient group $\mathrm{GL}_n(F)/\mathrm{Im}(\nu_n)$

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1. Find the order of the group $\mathrm{GL}_n(F)$ of invertible $n \times n$ matrices with entries in a field F having q elements.

2. Find a subgroup H of $\mathrm{GL}_n(F)$ such that $\mathrm{GL}_n(F)$ is isomorphic to the semi-direct product of H with $\mathrm{SL}_n(F)$ for the action of the former (by conjugation within $\mathrm{GL}_n(F)$) on the latter.

3. Let F be a field, and let α denote the action by fractional linear transformations of $\mathrm{GL}_2(F)$ on the set $F \cup \{\infty\}$.
   (a) Show that α is a transitive action.
   (b) What is the isotropy group at ∞?
   (c) Explain briefly why α is induced by an action of $\mathrm{PGL}_2(F)$.

4. Let $\mathbf{F}_3$ denote the field of 3 elements. Observe that the order of the group $\mathrm{GL}_2(\mathbf{F}_3)$ is 48 and that the groups $\mathrm{SL}_2(\mathbf{F}_3)$ and $\mathrm{PGL}_2(\mathbf{F}_3)$ both have order 24. The group S_4 of all permutations of a set of 4 elements also has order 24. Determine which, if any, of these three groups of order 24 are isomorphic.

5. Let F be a field, and let V be a finite-dimensional vector space over F. V^* will denote the dual space of V. The Heisenberg group $\mathrm{Hs}(V)$ is the set $V \times V^* \times F$ with group law given by

$$(v_1, f_1, t_1) * (v_2, f_2, t_2) \quad = \quad (v_1 + v_2,\ f_1 + f_2,\ t_1 + t_2 + f_2(v_1)) \ .$$

   (a) Show that the center C of $\mathrm{Hs}(V)$ is the subgroup $\{0\} \times \{0\} \times F$.
   (b) Explain why there is no subgroup H of $\mathrm{Hs}(V)$ such that $\mathrm{Hs}(V)$ is a semi-direct product of H and C.
   (c) What action by automorphisms of V^* on the direct product $V \times F$ gives rise to a semi-direct product of V^* and $V \times F$ that is isomorphic to $\mathrm{Hs}(V)$?