Math 502 Class Slides

February 5, 2008

1 Exercise 63:6 on floating-point approximation

- Approximate $e^{\pi\sqrt{163}} - 262537412640768744$ with precision settings of 15, 25, and 35 digits
  - The given real number is close to zero; it is initially unclear whether it is positive or negative.
  - Maple will hold $e^{\pi\sqrt{163}}$ symbolically unless floating point conversion is forced in some way.
  - Maple's rounding function *round()* does not force floating point conversion.
  - Results obtained using floating point arithmetic (computerized emulation of real number arithmetic) may vary. Control of precision is, therefore, important.
  - Floating point calculations below do not match those usually obtained on machines in the classroom.
  - Using `x-n` below with *evalf()* is not significantly different from repeatedly using `exp(Pi*sqrt(163))-262537412640768744`.

```
> x:=exp(Pi*sqrt(163));
                                            1/2
                              x := exp(Pi 163   )

> round(x);
                                            1/2
                             round(exp(Pi 163   ))

> n:=round(evalf(x,50));
                            n := 262537412640768744

> evalf(x-n,15);
                                       0.

> evalf(x-n,25);
                                       0.

> evalf(x-n,35);
                                         -12
                               -0.74993 10
```

2 Continued Fraction Expansion of a Real Number

For a real number x its continued fraction expansion is a sequence $\mathrm{CF}(x) = [a_0, a_1, a_2, \ldots]$, having finite or infinite length, of integers a_j. Usually it is assumed that $a_j > 0$ for $j > 0$. How is $\mathrm{CF}(x)$ defined?

First, one observes that x may be written uniquely in the form $x = n + t$ where n is an integer and $0 \le t < 1$. One defines

$$\mathrm{integerPart}(x) = \mathrm{floor}(x) = n$$

and

$$\mathrm{fracPart}(x) = \mathrm{modOne}(x) = t \quad .$$

If $x = n$ is an integer, then $\mathrm{CF}(x) = [n]$ is an integer sequence of length one whose sole entry is x. If x is not an integer, then $t = \mathrm{modOne}(x) > 0$, and $\mathrm{CF}(x)$ is defined recursively by pre-pending $n = \mathrm{floor}(x)$ to the sequence $\mathrm{CF}(1/t)$.

If $[a_0, a_1, a_2, \ldots]$ turns out to be a finite sequence $[a_0, a_1, a_2, \ldots, a_k]$, then

$$x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots + \frac{1}{a_k}}} ,$$

and, therefore, x must be a rational number (i.e., x must be the quotient of two integers). Conversely if x is rational, then by the Euclidean algorithm[1] (applied to its numerator and denominator), its continued fraction must have finite length.

When $\mathrm{CF}(x) = [a_0, a_1, a_2, \ldots]$, one sometimes writes

$$x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\cdots}}} ,$$

and in the case of a sequence $[a_0, a_1, a_2, \ldots]$ of infinite length one has

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\cdots}}} = \lim_{k \to \infty} \left(a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots + \frac{1}{a_k}}} \right) \quad .$$

When $a_j > 0$ for all $j > 0$, the sequence of "partial" continued fractions, which are called the *convergents*[2] of the continued fraction, is guaranteed to be a convergent sequence, i.e., the limit above always exists.

For more see Wikipedia: `http://en.wikipedia.org/wiki/Continued_fraction`

[1] URI: http://en.wikipedia.org/wiki/Euclidean_algorithm
[2] Reference target for key: "cvgts"

3 Examples of Continued Fractions

1. $$\frac{25}{7} = [3,1,1,3] = 3 + \frac{1}{1+\frac{1}{1+\frac{1}{3}}}$$

2. $$\frac{1+\sqrt{5}}{2} = [1,1,1,1,\dots] = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{\dots}}}}$$

4 Continued Fractions in Maple

Maple has at least two functions for generating continued fractions:

- The *confrac* regime in the *convert* facility:

```
> convert(25/7,confrac);
                                          [3, 1, 1, 3]
```

- The function *cfrac* in the *numtheory* package:

```
> numtheory[cfrac](25/7);
                                               1
                                     3 + -----------
                                                  1
                                         1 + -------
                                             1 + 1/3

> numtheory[cfrac](exp(1), 10, quotients);
                          [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1,
...]
```

5 Exercise 64:9 on Rational Approximation of π

- The convergents of the continued fraction expansion of a real number are known to provide "best rational approximations" to the real number in a certain sense that may be made precise. (See, for example, *Wikipedia*[3].)
- Convergents may be obtained with a call to the *confrac* regime of the *convert* facility by supplying a third argument for the length of the desired initial segment of the continued fraction expansion and a fourth argument for the user-supplied (and case sensitive) name of a variable in which to store the corresponding convergents.
-
```
> convert(Pi,confrac, 10, 'partialEvals');
                        [3, 7, 15, 1, 292, 1, 1, 1, 2, 1]

> partialEvals;
               333  355  103993  104348  208341  312689  833719  1146408
     [3, 22/7, ---, ---, ------, ------, ------, ------, ------, -------]
               106  113  33102   33215   66317   99532   265381  364913
```

- Conclude that the best rational approximation to π of the kind provided by the convergents of its continued fraction expansion, with denominator smaller than 1000, is

$$\frac{355}{113} \quad .$$

[3]URI: http://en.wikipedia.org/wiki/Continued_fraction#Best_rational_approximations