Advanced Linear Algebra (Math 424/524)

Handout on Matrices and Bases

If $\mathbf{v} = (v_1 \ldots v_n)$ is a row of vectors, each from a vector space V, and if x is a column of scalars of length n, the product $\mathbf{v}x$ of the row of vectors with the column of scalars may be regarded as the *pseudo matrix multiplication* that evaluates to $v_1x_1 + \ldots + v_nx_n$, which is an element of V.

The vectors in $\mathbf{v}$ span V if every vector in V may be written as $\mathbf{v}x$ for a least one column x. The vectors in $\mathbf{v}$ are linearly independent if a vector in V may be written in the form $\mathbf{v}x$ in at most one way. Finally, the vectors in $\mathbf{v}$ form a basis of V if and only if every vector in $\mathbf{v}$ may be written in the form $\mathbf{v}x$ in exactly one way. One calls x the *column of coordinates* of the vector $\mathbf{v}x$ relative to the basis $\mathbf{v}$.

If $\mathbf{v}$ is a basis of V and $\mathbf{v}'$ is another basis of V, then one has $v'_j = \mathbf{v}a_j$ for exactly one column a_j, and the $n \times n$ matrix $a = (a_1 \ldots a_n)$ is invertible. One may write the pseudo matrix relations

$$\mathbf{v}' = \mathbf{v}\,A \quad \text{and} \quad \mathbf{v} = \mathbf{v}'\,A^{-1} \quad .$$

In this situation either A or its inverse may be referred to as a *matrix of change of basis*. Neither should be viewed as the matrix of a linear map. In this context the expressions $\mathbf{v}x$ and $\mathbf{v}'x'$ represent the same vector in V if and only if the columns x and x' are related by the formula $x' = A^{-1}x$.

These and similar pseudo matrix formulas are characterized by the property that if all abstract vectors are replaced by their columns of coordinates relative to some (related or not) basis, then the relation becomes a regular matrix formula.

If f is a linear map from a vector space V to a vector space W, $\mathbf{v}$ a basis of V, and $\mathbf{w}$ a basis of W, then one says that M is the matrix of f relative to $\mathbf{v}$ and $\mathbf{w}$ if one has $y = Mx$ whenever x is the column of coordinates of a vector v in V relative to the basis $\mathbf{v}$ and y is the column of coordinates of $f(v)$ relative to the basis $\mathbf{w}$.

There is a straightforward way of translating the previous description of the *matrix of a linear map relative to given bases* into the language of bases rather than, as above, the language of coordinates. For this one uses the notation $f(\mathbf{v})$ to denote the row of vectors $f(v_j)$ in W. The row $f(\mathbf{v})$ need not be a basis of W. Since, however, $\mathbf{w}$ is a basis of W, one may write, for each subscript j, $f(v_j) = \mathbf{w}M_j$ for a unique column M_j of scalars. Then the matrix of f is the matrix $M = (M_1 \ldots M_n)$, and, indeed, one has the equivalent pseudo matrix formula $f(\mathbf{v}) = \mathbf{w}M$.

What happens to the matrix of a linear map when the bases are changed? Suppose $\mathbf{v}' = \mathbf{v}A$, $\mathbf{w}' = \mathbf{w}B$, $f(\mathbf{v}) = \mathbf{w}M$, and $f(\mathbf{v})' = \mathbf{w}'M'$. Since f is linear, one sees that the first of these formulas leads to the relation $f(\mathbf{v}') = f(\mathbf{v})A$. It then follows by elementary matrix arithmetic that $M' = B^{-1}MA$.

When f is a linear map from a vector space to itself, one usually uses a single basis in the roles of both $\mathbf{v}$ and $\mathbf{w}$ above.

When f is a linear map from a vector space V to itself, M its matrix relative to a basis $\mathbf{v}$, and M' its matrix relative to a basis $\mathbf{v}'$ consisting of characteristic vectors of f (if such a basis exists), then M' is a matrix whose diagonal entries are the proper values of f and one has $M' = A^{-1}MA$ where $\mathbf{v}' = \mathbf{v}A$.

Matrices will be used in three ways in this course:

1. The matrix of a linear map relative to given bases of the domain and the target furnishes a definition of the linear map in terms of the given bases.
2. The matrix of change of basis for two given bases defines the relationship between the two bases.
3. The matrix of a *bilinear form* on the product of two vector spaces relative to given bases of those vector spaces characterizes the bilinear form relative to those bases. This use of matrices, combined with the *Principal Axis Theorem*, underlies the normal form of a conic section that is usually presented in a U.S. standard first year calculus sequence.