Transformation Geometry — Math 331

April 28, 2004

Transformation Groups: IV

We know that building blocks for isometries of $\mathbf{R}^2$ are the reflections in lines in $\mathbf{R}^2$ and for isometries of $\mathbf{R}^3$ are the mirror reflections in planes in $\mathbf{R}^3$. Every isometry is the product of a finite number of these building blocks. What is a minimal set of additional building blocks that can be used to form every affine transformation? Two theorems from linear algebra are useful.

Theorem. (*The Principal Axis Theorem*) Every symmetric matrix is conjugate by an orthogonal matrix to a diagonal matrix.

Theorem. Every invertible matrix is the product of a symmetric matrix with positive eigenvalues and an orthogonal matrix.

Proof. The principal axis theorem is standard content of second undergraduate year linear algebra. For the second theorem let A be an invertible $n \times n$ matrix. Let T be the matrix $A\,{}^tA$. Clearly, T is symmetric, and, therefore, by the principal axis theorem $T = VC\,{}^tV$ with V orthogonal and C diagonal. If λ is an eigenvalue of T and x an eigenvector for λ, then $Tx = \lambda x \neq 0$ since A is invertible, and

$$\lambda\,||x||^2 \;=\; {}^tx(\lambda x) \;=\; {}^txTx \;=\; {}^txA\,{}^tAx \;=\; {}^t({}^tAx)({}^tAx) \;=\; \left|\left|{}^tAx\right|\right|^2 > 0$$

and, therefore, $\lambda > 0$. Thus, all of the diagonal elements in the diagonal matrix C are positive, and so $C = D^2$ is the square of a diagonal matrix with positive diagonal entries. Let S be the symmetric matrix $S = VD\,{}^tV$. Then one may check that $U = S^{-1}A$ is an orthogonal matrix, and so $A = SU$.

Definition. If r is a real number and a a point of $\mathbf{R}^n$, the affine transformation $D_r(a)$ of $\mathbf{R}^n$ defined by $(D_r(a))(x) = (1-r)a + rx = a + r(x-a)$ is called a dilatation. The point a, which is always a fixed point, is called the *center* of $D_r(a)$. Unless there is mention to the contrary the term *dilatation* will be understood to include only the case $r > 0$.

If u is a given vector in $\mathbf{R}^n$, recall that any vector v may be written as $v' + v''$ where v' is parallel to u and v'' is perpendicular to u. One sometimes writes $v' = \mathrm{proj}_u(v)$ for the parallel component of v and $v'' = \mathrm{perp}_u(v)$ for the perpendicular component of v relative to u.

Definition. A 1-dimensional dilatation of $\mathbf{R}^n$ is an affine transformation of the form

$$(\delta_r(a,u))(x) \;=\; a + r\,\mathrm{proj}_u(x-a) + \mathrm{perp}_u(x-a)$$

for some point a, some vector u, and some scalar $r > 0$.

Corollary. Every affine transformation of $\mathbf{R}^n$ may be formed as a finite product of isometries and 1-dimensional dilatations.

Proof. Let $f(x) = Ax + v$. Since translation by v is an isometry, it is a question about the affine transformation $x \mapsto Ax$. In view of the decomposition $A = SU$ with S symmetric having positive eigenvalues and U orthogonal, it is then a question about the transformation $x \mapsto Sx$, and by the principal axis theorem then a question about $x \mapsto Dx$ where D is a diagonal matrix with positive entries. Clearly this last affine transformation is a product of 1-dimensional dilatations.

Assignment for Friday, April 30

1. Represent the affine transformation $x \mapsto Ax$ of $\mathbf{R}^2$ as a product of isometries and 1-dimensional dilatations when
$$A \;=\; \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} .$$

2. Represent the affine transformation $x \mapsto Ax$ of $\mathbf{R}^3$ as a product of isometries and 1-dimensional dilatations when
$$A \;=\; \begin{pmatrix} 6 & -3 & 6 \\ -1 & 2 & 2 \\ 4 & 4 & -2 \end{pmatrix} .$$