Transformation Geometry — Math 331

April 26, 2004

Transformation Groups: III

Definition. If G is a group of transformations of a set X, g an element of G, S a subset of X, and $g(S)$ the set to which S is carried by g, one says that g **stabilizes** S if $g(S) = S$. The set of all transformations g in G that stabilize S is called the **stabilizer of** S **in** G. A subset F of G **stabilizes** S if it is contained in the stabilizer of S in G, i.e., if $g(S) = S$ for every g in F.

Example. If γ is a glide reflection of the plane $\mathbf{R}^2$, l its axis, and G the set of isometries of the form γ^k for $k = 0, \pm 1, \pm 2, \ldots$, then G is a group that stabilizes l and is a proper subgroup of the stabilizer of l in the group of all isometries of $\mathbf{R}^2$.

Proposition 1. For any subset S of X the stabilizer of S in G is a subgroup of G.

Proposition 2. If G is a group of transformations of X, S a subset of X, g an element of G, $T = g(S)$, and H the stabilizer of S in G, then the stabilizer of T in G is the conjugate of H by g, i.e., the subgroup

$$gHg^{-1} = \left\{ ghg^{-1} \mid h \in H \right\} .$$

Definition. If G is a group and H a subgroup of G, H is normal in G if $gHg^{-1} = H$ for every g in G.

Definition. If G is a group of transformations of a set X and S a subset of X that is stabilized by G, one says that G is **transitive on** S if for each pair x, y of points of S there is at least one element g in G such that $y = g(x)$. G is **simply transitive on** S if for each pair x, y of points of S there is exactly one element g in G such that $y = g(x)$.

Example. The isotropy group at the origin in the group of isometries of $\mathbf{R}^2$ stabilizes the unit circle and is transitive on the unit circle but not simple transitive on the unit circle.

Example. The group of translations of $\mathbf{R}^n$ is simply transitive on $\mathbf{R}^n$.

Assignment for Wednesday, April 28

1. If G is a group of transformations of a set X and S is a subset of X that is stabilized by G, does it then follow that G is a group of transformations of S?
2. What subsets of $\mathbf{R}^3$ are stabilized by the isotropy group of the Euclidean group at the origin?
3. What is the stabilizer of the first coordinate axis in the group of all isometries of $\mathbf{R}^2$?
4. Prove Propostion 1.
5. Prove Propostion 2.
6. Let G be a group of transformations of a set X with G transitive on X, and let x_0 be a given point of X. Prove that if the isotropy group of G at x_0 is a normal subgroup of G, then it consists of only the identity transformation of X.
7. Show that the group of translations of $\mathbf{R}^n$ is normal in both the isometry group of degree n and the affine group of degree n, but that the isometry group of degree n is not normal in the affine group of degree n.