Transformation Geometry — Math 331

April 23, 2004

Transformation Groups: II

Definition. The **affine group** of degree n is the transformation group consisting of all affine transformations of $\mathbf{R}^n$.

Definition. The **Euclidean group** of degree n is the transformation group consisting of all isometries of $\mathbf{R}^n$. The term **isometry group** is also used.

Definition. If G is a transformation group, a subset H of G is a **subgroup** of G if H is itself a transformation group.

Example. For a given integer $n \geq 1$ the Euclidean group of degree n is a subgroup of the affine group of degree n.

Definition. If G is a group of transformations of a set X and S is a subset of X, the set G_S of all transformations g in G that fix every point of the set S is called the **fixer of** S **in** G.

Proposition 1. For any subset S of X the set G_S is a subgroup of G.

Definition. If $S = \{x\}$ is the set consisting of a single point x of X, $G_S = G_x$ is called the **isotropy subgroup of** G **at** x.

Proposition 2. If G is a group of transformations of X, S a subset of X, g an element of G, and $T = g(S)$ the set to which S is carried by g, then one has

$$G_T \;=\; gG_Sg^{-1} \;=\; \left\{gfg^{-1} \mid f \in G_S\right\} \quad .$$

Assignment for Monday, April 26

1. What is the isotropy subgroup of the affine group at the origin of $\mathbf{R}^n$?
2. If A is the third coordinate axis in $\mathbf{R}^3$, what is the fixer G_A of A in the affine group G of degree 3?
3. With the point $(x, y) \in \mathbf{R}^2$ corresponding to the point with homogeneous coordinates $(x, y, 1-x-y)$ in $\mathbf{P}^2$ and $x + y + z = 0$ the equation of the line l_∞ at infinity in $\mathbf{P}^2$, the affine group G of degree 2 is realized as a subgroup of the group of all projective transformations of $\mathbf{P}^2$. What is the fixer G_{l_∞}?
4. Write a proof of proposition 1.
5. Write a proof of proposition 2.
6. The composition of the rotation of $\mathbf{R}^3$ by the angle π about the line through $(0, 0, 0)$ and $(1, 1, 1)$ following the mirror reflection in the plane normal to the vector $(0, 0, 1)$ is a reflective rotation (though the axis of the given rotation is not perpendicular to the fixed plane of the given reflection). For this reflective rotation find its axis of rotation, the fixed plane of its reflection component (which is normal to the axis of rotation), and the cosine of the angle of rotation about its axis.