Transformation Geometry — Math 331

April 21, 2004

The Notion of Transformation Group

Recall that a transformation of a set is an *invertible* map from that set to itself.

Definition. A collection of transformations of a set is a **transformation group** if

1. It contains the identity transformation.
2. The inverse of any transformation in the set is also in the set.
3. The composition of any pair of transformations in the set is in the set.

Examples of transformation groups.

1. The set of all transformations of any set.
2. The set of all affine transformations of $\mathbf{R}^n$.
3. The set of all isometries of $\mathbf{R}^n$.
4. The set of all translations of $\mathbf{R}^n$.

The notion of transformation group is a special case of the general concept of (abstract) group.

Definition. A group is a set G endowed with an operation defined for every pair of elements x, y in G that yields an element $x * y$ in G such that the following rules hold:

1. $(x * y) * z = x * (y * z)$.
2. There is an element e in G such that for each element x in G (a) the relation $x*e = e*x = x$ holds and (b) there is an element x' in G such that the relation $x * x' = x' * x = e$ holds.

Example. Any transformation group G is a group when for any pair f, g of transformations $f * g = f \circ g$, the composition of f and g, the "group identity" e is the identity transformation, and the "group inverse" f' for a transformation f is the inverse transformation f^{-1}.

Example. The set $\mathrm{GL}_n(\mathbf{R})$ of invertible $n \times n$ matrices is a group when $M * N$ is the matrix product of M and N, the "group identity" is the identity matrix, and the "group inverse" M' of a matrix M is the matrix inverse M^{-1}.

Assignment for Friday, April 23

1. Which classes of isometries of $\mathbf{R}^3$ form transformation groups?
2. Which unions of classes of isometries of $\mathbf{R}^3$ form transformation groups?
3. What group of transformations of $\mathbf{R}^3$ admits an obvious bijective correspondence with the group $\mathrm{GL}_3(\mathbf{R})$ having the property that matrix multiplication corresponds to composition of transformations?
4. What abstract group admits an obvious bijective correspondence with the transformation group consisting of all of the translations of $\mathbf{R}^n$ having the property that the group operation $*$ corresponds to composition of translations?