Transformation Geometry — Math 331

April 12, 2004

The Classification of Isometries of R^3: II

The key to analyzing an isometry $f(x) = Ux + v$ of $\mathbf{R}^3$ is to begin by analyzing the fixed lines and planes through 0 for the linear map given by the 3×3 orthogonal matrix U. This rests on looking at the eigenvectors of U. Let $u = \det(U) = \pm 1$. The three eigenvalues of U (allowing for possible multiple eigenvalues) are u and $\cos\theta \pm i \sin\theta$ for some θ, $0 \leq \theta < 2\pi$.

Definition. A **rotation** of $\mathbf{R}^3$ is an orientation-preserving isometry that admits at least one fixed point.

Proposition. A non-identity rotation of $\mathbf{R}^3$ has a line of fixed points (called its **axis**). If f is a rotation, then the restriction of f to any plane orthogonal to its axis is the rotation of that plane about the point where it meets the axis by an angle that does not depend on the orthogonal plane.

Proof. If f is a rotation, then f may be conjugated by the translation carrying the origin to its fixed point in order to obtain an orientation-preserving isometry that fixes the origin. Thus, it suffices to prove the proposition when the fixed point is the origin, i.e., for the linear transformation given by an orthogonal matrix. Since f is orientation-preserving, $\det(U) = 1$, and, therefore 1 is an eigenvalue of U. If v is an eigenvector of U for the eigenvalue 1, then every point tv is a fixed point of f, and, because f is angle-preserving, it carries any plane perpendicular to v to itself, giving rise to an orientation-preserving isometry of that plane with a fixed point, i.e., a rotation of that plane.

Definition. The term **point reflection** is used for an isometry of $\mathbf{R}^3$ having the form $x \mapsto 2c - x$ in some, hence *any* (since the combination of c and x is barycentric), affine coordinate system.

Definition. A **mirror reflection** (sometimes simply *reflection*) is an isometry of $\mathbf{R}^3$ having precisely a plane of fixed points.

Proposition. The point reflections and mirror reflections of $\mathbf{R}^3$ are orientation-reversing.

Proof. It is obvious for the case of a point reflection. If an isometry has precisely a plane of fixed points, it is sufficient to assume that the plane of fixed points passes through the origin and, therefore $f(x) = Ux$ where U is orthogonal. Since f is angle-preserving, the line through the origin normal to the fixed plane must be stabilized by f. However, if that line is fixed by f, then f fixes each of the three vectors of $\mathbf{R}^3$ in some orthonormal basis of $\mathbf{R}^3$, and then f would be the identity. The only other alternative is that the normal line is a line of eigenvectors for some eigenvalue other than 1, and the only possible other eigenvalue is -1. Therefore, $\det(U) = -1$, and f is orientation-reversing.

Proposition. There is one and only one mirror reflection with a given fixed plane. If Π is a plane, and p is the orthogonal projection of $\mathbf{R}^3$ on Π, then the unique mirror reflection f in Π is given in any affine coordinate system by the formula $f(x) = 2p(x) - x$ (a barycentric combination of the points x and $p(x)$).

Proof. This follows from the description of a mirror reflection obtained in the proof of the preceding propostion.

Assignment for Wednesday, April 14

1. Give specific geometric descriptions of the isometries $f(x) = Ux + v$ of $\mathbf{R}^3$ given by the following pairs of 3×3 orthogonal matrices U and vectors v.

   (a)
   $$U = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad v = 0 \quad .$$

   (b)
   $$U = \begin{pmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{pmatrix}, \quad v = 0 \quad .$$

   (c)
   $$U = \begin{pmatrix} 2/3 & -1/3 & 2/3 \\ -1/3 & 2/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{pmatrix}, \quad v = 0 \quad .$$

2. Prove that an orientation-reversing isometry with at least two fixed points must be a mirror reflection.