Transformation Geometry — Math 331

April 2, 2004

The Classification of Isometries of R^3: I

- Recall that every isometry of $\mathbf{R}^n$ is the composition of a translation and an isometry that fixes the origin, and that every isometry fixing the origin has the form $x \mapsto Ux$ where U is an orthogonal matrix.
- There is a four-way division according to (a) whether an isometry is orientation-preserving or not and (b) according to whether it has a fixed point or not. But for $n = 3$ this does not give a complete description.
- The key to understanding the geometric structure of the isometry given by a 3×3 orthogonal matrix is to understand its eigenvectors and eigenvalues. Once that is done, one needs to analyze the transformation that results when one of those is followed by a translation.
- The characteristic polynomial of an $n \times n$ matrix A is the determinant of the $n \times n$ matrix of polynomials $t1_n - A$ (with t the variable). It is a polynomial of degree n with leading coefficient 1 and constant term equal to $\det(-A) = (-1)^n \det(A)$.
- All of the eigenvalues of an orthogonal matrix must be of the form $a + ib$ where a and b are real with $a^2 + b^2 = 1$.
- Counting multiplicities, there are n complex roots of any polynomial f of degree $n \geq 1$. If the leading coefficient of the polynomial is 1, then the sum of its n complex roots is the negative of the coefficient of degree $n-1$, and the product of its n complex roots is the constant term multiplied by $(-1)^n$.
- Since the characteristic polynomial of an orthogonal matrix is a polynomial with real coefficients, any of its roots that are not real must occur in complex-conjugate pairs. The product of any two complex-conjugate eigenvalues of an orthogonal matrix must be 1.
- Since the degree of the characteristic polynomial of a 3×3 matrix is odd, at least one of the eigenvalues of a 3×3 matrix must be real.
- **Proposition.** The three eigenvalues, counting multiplicities, of a 3×3 orthogonal matrix must be one of 1 or -1 and both of $\cos\theta \pm i\sin\theta$ for some real value of θ, $0 \leq \theta < 2\pi$. If $\theta = 0$, then the latter two eigenvalues are both 1, and if $\theta = \pi$, then they are both -1.

Assignment for Monday, April 12

1. How may one describe the pencil of lines through the point $(3, -5, 2)$, (a point on the line at infinity represented by homogeneous coordinates relative to the affine basis $((1, 0), (0, 1), (0, 0))$) in terms of the ordinary Cartesian geometry of $\mathbf{R}^2$?
2. Let f denote the linear isometry of $\mathbf{R}^3$ given by the formula $f(x) = Mx$ where M is the 3×3 orthogonal matrix
$$M = \frac{1}{7}\begin{pmatrix} 6 & -2 & -3 \\ -2 & 3 & -6 \\ -3 & -6 & -2 \end{pmatrix} .$$
Describe f in geometric terms.
3. How much of the classification of the isometries of the plane would be obtained by pursuing a discussion for dimension 2 that is parallel to the discussion above for dimension 3?
4. Can a 3×3 orthogonal matrix other than the identity matrix be a scalar multiple of the affine matrix, relative to the affine basis $((1, 0), (0, 1), (0, 0))$, of an isometry of $\mathbf{R}^2$?