Transformation Geometry — Math 331

March 26, 2004

Projective Transformations of the Projective Plane

An affine transformation of the affine plane $\mathbf{R}^2$, via its affine matrix relative to a given affine basis, gives rise to a transformation of the projective plane $\mathbf{P}^2$. Because the affine matrix is an invertible 3×3 matrix, it gives rise to an invertible linear transformation of $\mathbf{R}^3$ and, therefore, an invertible map from $\mathbf{P}^2$ to itself. Such a map on $\mathbf{P}^2$ is an example of a projective transformation.

Definition. A *projective transformation* of $\mathbf{P}^2$ is an invertible map f from $\mathbf{P}^2$ to itself arising from some invertible 3×3 matrix M in the sense that if a point p of $\mathbf{P}^2$ is represented by a vector x of homogeneous coordinates in $\mathbf{R}^3$, then the point $f(p)$ in $\mathbf{P}^2$ is the point for which the vector Mx (with x regarded as a length 3 column) is a triple of homogeneous coordinates.

It is obvious that if both x and x' are vectors of homogeneous coordinates for a point p in $\mathbf{P}^2$, then Mx and Mx' represent the same point q in $\mathbf{P}^2$. Morever, if $M' = bM$ for some non-zero scalar $b \neq 0$, then $M'x = (bM)x = b(Mx)$, which is to say that if M' is a non-zero scalar multiple of M, then M and M' determine the same projective transformation f.

Proposition. An invertible 3×3 matrix M is the affine matrix of an affine transformation of $\mathbf{R}^2$ relative to a given affine basis of $\mathbf{R}^2$ if and only if each of its columns sums to the number 1.

Proof. By definition of the affine matrix of an affine transformation each column is a triple of barycentric coordinates and, hence, sums to 1. Conversely, given such an invertible matrix there is a unique affine transformation sending the members of the affine basis of $\mathbf{R}^2$ to the three barycentric combinations of the three points in the basis represented by the columns of the given matrix.

Theorem. A point p in $\mathbf{P}^2$ is a fixed point of the projective transformation f given by an invertible 3×3 matrix M if and only if a vector $x \neq 0$ of homogeneous coordinates of p is an eigenvector of M.

Proof. Regardless of the value of a non-zero scalar λ, the condition $Mx = \lambda x$ is precisely equivalent to $f(p) = p$.

Theorem. A line with homogeneous equation $a_1x_1 + a_2x_2 + a_3x_3 = 0$ (where $a = (a_1, a_2, a_3) \neq (0, 0, 0)$) in $\mathbf{P}^2$ is stabilized by the projective transformation f given by an invertible 3×3 matrix M if and only if its coefficient vector a is an eigenvector of the transpose of M.

Proof. Let l be the line. The point with homogeneous coordinate vector x lies on $f^{-1}(l)$ if and only if $f(x)$ lies on l if and only if ${}^t a\,(Mx) = 0$. Since ${}^t a\,(Mx) = ({}^t aM)\,x = {}^t\left({}^t Ma\right)x$, the line $f^{-1}(l)$ has the equation $b_1x_1 + b_2x_2 + b_3x_3 = 0$ where $b = {}^t Ma$, and, therefore, $f(l) = l$ if and only if $f^{-1}(l) = l$ if and only if a and b are parallel vectors, i.e., a is an eigenvector of ${}^t M$.

Corollary. The line at infinity, i.e., the line $x_1 + x_2 + x_3 = 0$, is stabilized by the projective transformation given by M if and only if M is a non-zero scalar multiple of the affine matrix of an affine transformation.

Proof. By the theorem the line at infinity is stabilized if and only if the coefficient vector $(1, 1, 1)$ is an eigenvector of ${}^t M$. If $\lambda \neq 0$ is the corresponding eigenvalue of ${}^t M$, then this condition is equivalent to the statement that each column of $\lambda^{-1}M$ sums to 1.

Exercises due Monday, March 29

1. Find the affine matrix of the translation of $\mathbf{R}^2$ by the vector $(3, 4)$, and then find all fixed points and all stabilized lines of the corresponding projective transformation.

2. Repeat the previous exercise for the glide reflection f of $\mathbf{R}^2$ given by

$$f(x) \;=\; \frac{1}{13}\begin{pmatrix} 12 & 5 \\ 5 & -12 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 6 \\ -4 \end{pmatrix} \quad .$$

3. Give an example of an invertible 3×3 matrix for which the corresponding projective transformation carries the line at infinity to the line $z = 0$.

4. What specific restriction on a 3×3 matrix is imposed by the condition that it stabilizes the line in $\mathbf{P}^2$ that has the homogeneous equation $z = 0$?

5. Why must every projective transformation of $\mathbf{P}^2$ have at least one fixed point and one stabilized line?