Transformation Geometry — Math 331

March 12, 2004

The Affine Matrix of an Affine Transformation

Recall from the study of linear algebra that if f is a linear map from $\mathbf{R}^n$ to itself and $\mathbf{v} = \{v_1, \ldots, v_n\}$ is a linear basis of $\mathbf{R}^n$, then the *matrix of f with respect to the basis* $\mathbf{v}$ is the $n \times n$ matrix M whose j-th column, for $1 \leq j \leq n$, is the column of coordinates of $f(v_j)$ relative to $\mathbf{v}$, i.e.,

$$f(v_j) \;=\; \sum_{i=1}^{n} M_{ij} v_i \,,\; 1 \leq j \leq n \quad .$$

Definition. If $\mathbf{p} = \{p_0, \ldots, p_n\}$ is an affine basis of $\mathbf{R}^n$ and f is an affine map from $\mathbf{R}^n$ to itself then the *affine matrix of f with respect to the affine basis* $\mathbf{p}$ is the $(n+1) \times (n+1)$ matrix M whose j-th column, for $0 \leq j \leq n$, is the column of barycentric coordinates of $f(p_j)$ relative to $\mathbf{p}$, i.e.,

$$f(p_j) \;=\; \sum_{i=0}^{n} M_{ij} p_i \;\text{ with }\; \sum_{i=0}^{n} M_{ij} \;=\; 1 \,,\; 0 \leq j \leq n \quad .$$

Proposition. If p is a point of $\mathbf{R}^n$ having barycentric coordinates $(x_0, \ldots, x_n)$ relative to the affine basis $\mathbf{p}$ and if f is an affine map having matrix M relative to $\mathbf{p}$, then $f(p)$ is the point of $\mathbf{R}^n$ having barycentric coordinates $(y_0, \ldots, y_n)$ relative to $\mathbf{p}$ where the vectors x and y, when regarded as columns, are related by the formula $y = Mx$.

Proof. Because f preserves barycentric combinations and $p = x_0 p_0 + \ldots x_n p_n$ with $x_0 + \ldots + x_n = 1$, it follows that

$$\begin{aligned} f(p) &= \sum_j x_j f(p_j) \;=\; \sum_j x_j \left(\sum_i M_{ij} p_i \right) \\ &= \sum_{ij} M_{ij} x_j p_i \;=\; \sum_i \left(\sum_j M_{ij} x_j \right) p_i \\ &= \sum_i y_i p_i \text{ where } y \;=\; Mx \end{aligned}$$

One needs to check that the last line is indeed a barycentric combination of the p_i, i.e., that $y_0 + \ldots + y_n = 1$. This follows from the fact that y is the x-barycentric combination of the (weight 1) columns of M

Exercises due Monday, March 15

1. Show that the map $\varphi : \mathbf{R}^2 \to \mathbf{R}^3$ given by $(x_1, x_2) \mapsto (x_1, x_2, 1 - x_1 - x_2)$ is an affine map.
2. Conclude from the first exercise that if τ is translation of $\mathbf{R}^2$ by the vector $a = (a_1, a_2)$, then $\varphi(\tau(x)) = \varphi(x) + \tilde{a}$ where $\tilde{a}$ is the weight 0 triple (a_1, a_2, a_3) with $a_3 = -a_1 - a_2$.
3. *(Continuing)* Find the affine matrix of the translation τ.
4. Find the affine matrix of the half turn of $\mathbf{R}^2$ about the point c, i.e., the affine transformation $x \mapsto 2c - x$.
5. Show that if M is the affine matrix of the affine transformation $f(x) = Ux + v$ of $\mathbf{R}^2$, then $\det M = \det U$.