Transformation Geometry — Math 331

March 17, 2004

Exercises on Lines Stabilized by an Affine Transformation

Several exercises in the assignment due March 8 deal with applications of the following corollary given there:

> **Corollary.** A line $a \cdot x = c$ (with $a \neq (0,0)$) in $\mathbf{R}^2$ is stabilized by the affine transformation $f(x) = Ux + v$ if and only if there is a non-zero scalar λ such that ${}^tUa = \lambda a$ and $a \cdot v = (1-\lambda)c$.

The basic idea in applying this theorem to find a stabilized line is to observe, first of all, that the coefficient vector a of the equation is normal to the line and, therefore, determines the parallel class of the line. The lines in a parallel class are distinguished by different values of c for a given vector a. The corollary states that a must be an eigenvector of the matrix ${}^tU$, and unless the corresponding eigenvalue is 1, the scalar c is determined by a through the formula $c = (1-\lambda)^{-1} a \cdot v$.

For example, if 1 is not an eigenvalue of U (which has the same eigenvalues as ${}^tU$ though not the same eigenvectors), then there is one and only one line stabilized by f for each parallel class of eigenvectors a of ${}^tU$. When U has 1 as an eigenvalue, there will still be a unique stabilized line normal to the eigenvector of ${}^tU$ for an eigenvalue different from 1 if there is one.

There will be a stabilized line normal to an eigenvector a of ${}^tU$ for the eigenvalue 1 if and only if a is normal to the vector v. In this case the stabilized line, which is normal to a, must be parallel to v, and every line parallel to v is stabilized by f.

A fixed point c of f is characterized by the equation $(1-U)c = v$, and, therefore, there is a unique fixed point — hence no line that is fixed — when 1 is not an eigenvalue of U. If, on the other hand, 1 is an eigenvalue of U but U is not the identity matrix, then $1-U$ is a rank 1 matrix and there is a line of fixed points c of f if and only if v is in the column space of $1-U$.

Two of the exercises in question were:

> **4.** Can an orientation-reversing order 2 affine transformation of the plane stabilize a line it does not fix?
>
> **5.** Apply the corollary above to determine all lines stabilized by a glide reflection.

In both cases the matrix U must be a matrix of order 2 with $\det U < 0$. Since $U^2 = 1$, $(\det U)^2 = 1$, and, therefore, $\det U = -1$. Since $(1+U)(1-U) = (1-U)(1+U) = 1-U^2 = 0$, it follows that both $1+U$ and $1-U$ are rank 1 matrices since neither is invertible (and neither is 0 lest it be ± 1 and so have determinant 1). Therefore, the eigenspace of U for the eigenvalue 1, which is the nullspace of $1-U$, is the column space of $1+U$, and that for the eigenvalue -1, which is the nullspace of the matrix $1+U$, is the column space of the matrix $1-U$. Moreover, all of the statements of this paragraph continue to hold if U is replaced by ${}^tU$. Finally, the column space of $1-{}^tU$ is orthogonal to that of $1+U$, and the column space of $1+{}^tU$ is orthogonal to that of $1-U$.

In the case of a glide reflection (No. 5) U is a symmetric orthogonal matrix, the axis of f is parallel to the eigenspace of U for the eigenvalue 1, and a coefficient vector a for the equation of the axis is normal to the axis, hence, in the eigenspace for the eigenvalue -1 of $U = {}^tU$, and, therefore, the axis of the glide reflection is the unique stabilized line with coefficient vector a in the eigenspace for the eigenvalue -1. If there is a stabilized line for which the coefficient vector a is an eigenvector for the eigenvalue 1 of ${}^tU = U$, then by the corollary above v must be perpendicular to a, hence, perpendicular to the eigenspace of U for the eigenvalue 1, and so f would be a reflection, not a glide reflection. Therefore, the axis of a glide reflection is the only line stabilized by a glide reflection.

The case of an orientation-reversing affine transformation of order 2 (No. 4) contains the case of a reflection, in which case it is obvious that any line perpendicular to the axis of the reflection is stabilized. The general situation is somewhat similar.

Since f has order 2, one has $Uv = -v$, and, therefore, $(1+U)v = 0$, from which it follows that v is in the column space of $(1-U)$. Therefore, as above, there is a line of fixed points c of f that is a translate of the 1-dimensional nullspace of $1-U$. What is the equation of this line? In particular, what vector may be used as its coefficient (normal) vector? A vector parallel to the nullspace of $1-U$ is perpendicular to the row space of $1-U$. The row space of $1-U$ is the same as the column space of $1-{}^tU$, and the relation $(1+{}^tU)(1-{}^tU) = 0$ between two rank 1 matrices shows that the column space of $1-{}^tU$ is equal to the nullspace of $1+{}^tU$. Hence, a vector normal to the fixed line of f is characterized as an eigenvector for the eigenvalue -1 of ${}^tU$. It was explained above that there is one and only one stabilized line with this coefficient vector, and it is, therefore, the fixed line.

Of course, 1 is also an eigenvalue of ${}^tU$, and, as explained above, any line of the form $a \cdot x = c$ with a an eigenvector of ${}^tU$ for the eigenvalue 1 is a stabilized line provided that a is normal to v. This latter condition is seen as follows: from the fact that a is in the nullspace of $1-{}^tU$ it follows that a is normal to the row space of $1-{}^tU$, which is the same as the column space of $1-U$ and the nullspace of $1+U$ where v is known to reside. The stabilized lines $a \cdot x = c$, where ${}^tUa = a$, are not fixed lines since there is only one fixed line, as discussed above.