Transformation Geometry — Math 331

March 1, 2004

Reflections and Glide Reflections

An isometry is orientation-reversing if and only if relative to Cartesian coordinates it has the form $f(x) = Ux + v$ where U is a reflection matrix, i.e., an orthogonal matrix of determinant -1.

Proposition. If U is a 2×2 orthogonal matrix with determinant -1, then the linear transformation $\sigma(x) = Ux$ is the reflection in a line through the origin, and the following statements hold:

1. $1 - U^2 = (1+U)(1-U) = (1-U)(1+U) = 0$.
2. Any vector v of the form $(1+U)w$ for some vector w lies on the axis of σ.
3. Any vector v of the form $(1-U)w$ for some vector w is perpendicular to the axis of σ.

Proof. It is elementary (see the assignment due February 20) that U has the form

$$U = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$$

where $a^2+b^2 = 1$. Since both $U^{-1} = {}^tU$ and ${}^tU = U$, clearly $U^2 = 1$, and, therefore, $(1+U)(1-U) = (1-U)(1+U) = 0$. If $v = (1+U)w$, then $(1-U)v = (1-U)(1+U)w = 0$, hence, $Uv = v$, and, therefore, v lies on the axis of σ. If, on the other hand $v = (1-U)w$, then by similar reasoning $Uv = -v$, which characterizes vectors v perpendicular to the axis of σ.

Proposition. Let $f = Ux + v$ be an orientation-reversing isometry, and let

$$v' = \frac{1}{2}(1-U)\,v \quad \text{and} \quad v'' = \frac{1}{2}(1+U)\,v \quad .$$

Then f is the composition of the isometry $\sigma(x) = Ux + v'$, which is a reflection with axis parallel to the axis of the reflection $x \mapsto Ux$, followed by the translation $\tau(x) = x + v''$ by the vector v'', which is parallel to the axis of σ.

Proof. Clearly, $v' + v'' = v$, and, therefore, $f = \tau \circ \sigma$. Since v' is perpendicular to the axis of $x \mapsto Ux$, translation by v' is the composition of the reflections in two lines parallel to the axis of $x \mapsto Ux$, and one of those two lines may be chosen to be the axis of $x \mapsto Ux$ and then σ is seen to be the reflection in the other of the two parallel lines. From this follows:

Theorem Let $f = Ux + v$ be an orientation-reversing isometry. Then f is a reflection if and only if $Uv = -v$ and is a glide reflection otherwise.

Exercises due Wednesday, March 3

1. Let f be the affine transformation of the plane defined by

$$f(x) = \begin{pmatrix} 3/5 & 4/5 \\ 4/5 & -3/5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad .$$

   (a) What points x of the plane are "fixed" by f, i.e., satisfy $f(x) = x$?
   (b) What lines L in the plane are "stabilized" by f, i.e., satisfy the condition that $f(x)$ is on L if x is on L?
   (c) Find a reflection σ and a translation τ parallel to the axis of σ such that $f = \tau \circ \sigma$.

2. Explain briefly why the composition of any four reflections may always be written as the composition of two reflections.

3. Two triangles $\Delta\, ABC$ and $\Delta\, A'B'C'$ are, by definition, *congruent* if there is an isometry f for which the vertices of the one triangle are carried to the vertices of the other. If $\Delta\, ABC$ and $\Delta\, A'B'C'$ are congruent, then how many isometries f of the first with the second are there when

   (a) $\Delta\, ABC$ is equilateral.
   (b) The sides BC and CA have the same length, which is different from the length of AB.
   (c) No two sides of $\Delta\, ABC$ have the same length.