Transformation Geometry — Math 331

February 25, 2004

Discussion

- **Theorem.** Every orientation-preserving isometry of $\mathbf{R}^2$ without a fixed point is a translation.

  Proof. Let f be a given orientation-preserving isometry of $\mathbf{R}^2$. Since f is an affine transformation, $f(x) = Ux + v$ for some invertible matrix U and some vector v. Since f is an isometry, U is an orthogonal matrix and since f is orientation-preserving, $\det U = 1$ and, in fact, as in the argument for characterizing rotations,

  $$U = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

  where $a^2 + b^2 = 1$. A point x will be a fixed point of f if and only if $Ux + v = x$, or, equivalently, x is a solution of the linear system $(1 - U)x = v$. So f certainly has a fixed point when the matrix $1 - U$ is invertible, which is the case when its determinant $(1 - a)^2 + b^2 \neq 0$. Therefore, f can only fail to have a unique fixed point when $a = 1$ and $b = 0$, i.e., when $U = 1$ is the identity matrix and f, therefore, is a translation.

- **Theorem.** Every orientation-reversing isometry of $\mathbf{R}^2$ without a fixed point is a glide reflection.

  Proof. Let $f = Ux + v$ be a given orientation-reversing isometry without a fixed point. Since $\det U = -1$, as in the argument for characterizing reflections,

  $$U = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$$

  where $a^2+b^2 = 1$. With this condition on U the matrix $1-U$ has determinant 0 and, in fact, has rank 1. Therefore, from the nature of U alone the fixed point equation $(1 - U)x = v$ has a solution x if and only if v is in the one-dimensional column space of $1 - U$. Since it is given that f has no fixed point, the vector v is not in the column space of $1-U$, which means that $Uv \neq -v$. Let l be the line through the origin that is the axis of the reflection $x \mapsto Ux$. The condition obtained on v is equivalent to the statement that v is not perpendicular to l. Now decompose $v = v' + v''$ where v' is perpendicular to l and v'' is parallel to l. Then $f(x) = (Ux + v') + v''$, which shows that f is the composite obtained from translation by v'' following the reflection $x \mapsto Ux + v'$ having axis parallel to the line l. Therefore f meets the condition characterizing glide reflections.

Exercises due Friday, February 27

1. Let f be rotation about the point $(1, 0)$ through the angle $\pi/4$, and let g be rotation about the point $(0, 1)$ through the angle $\pi/6$. Show that $g \circ f$ is a rotation, and find its center and its angle of rotation.

2. When is it the case that the composition of two rotations about different centers is a rotation?

3. What type of isometry is the composition of a reflection with the half turn, i.e., rotation through the angle π, about a point not on the axis of the reflection?

4. If three lines l_1, l_2, l_3 intersect so as to form a triangle, what type of isometry is the composition $\sigma_3 \circ \sigma_2 \circ \sigma_1$ of the reflections in those lines?