Transformation Geometry — Math 331

February 13, 2004

Discussion

- We have studied four triples of lines associated with a given triangle $\Delta\, ABC$ having sides of lengths a, b, c and vertex angles α, β, γ. The following table, which is provided for reference, lists homogeneous coordinates relative to the vertices of the triangle for the intersection point P of each of four triples of coincident lines.

medians	$(1, 1, 1)$
angle bisectors	(a, b, c) or $(\sin\alpha, \sin\beta, \sin\gamma)$
altitudes	$(\tan\alpha, \tan\beta, \tan\gamma)$
perpendicular bisectors	$(\sin 2\alpha, \sin 2\beta, \sin 2\gamma)$

- Previously the point was made that homogeneous coordinates of the point where the altitudes of a triangle with acute angles intersect relative to the vertices of the triangle are (proportional to) the areas of the three subtriangles formed by that point as third vertex with any side of the given triangle. This is a special case of the more general:

  Theorem. Let A, B, and C be three non-collinear points, and let $P = uA + vB + wC$ $(u + v + w = 1)$ be a point inside $\Delta\, ABC$, i.e., with $u, v, w > 0$. Then u, v, w are, respectively, the ratios of the areas of $\Delta\, BCP$, $\Delta\, CAP$, $\Delta\, ABP$, respectively, to the area of $\Delta\, ABC$.

  Proof. By symmetry, it is enough to check that the area of $\Delta\, ABP$ is equal to w times the area of $\Delta\, ABC$. Since AB is a common side in these two triangles, it is enough to check that the altitude length from P to AB is w times the altitude length from C to AB. Let F be the point where the line CP meets AB, let S be the foot of the altitude from P to AB, and let R be the foot of the altitude from C to AB. Then $\Delta\, PSF$ is clearly similar to $\Delta\, CRF$. Therefore, the altitude length ratio $|PS| \,/\, |CR|$ is equal to the hypotenuse ratio $|PF| \,/\, |CF|$. By the principle of preservation of proportionality in barycentric coordinates $(u + v)F = uA + vB$. Hence, $P = (u + v)F + wC$, and by the fulcrum principle $|PF| \,/\, |CF| = w$.

Exercises due Wednesday, February 18

1. Let c, h, and q be given with $c > 0$ and $h > 0$. Let A, B, and C be the points in $\mathbf{R}^2$ defined by

$$A = (0, 0)\,,\ B = (c, 0)\,,\ \text{and}\ C = (q, h) \quad .$$

   (a) Show that A, B, C are not collinear.

   (b) Find the three points where the altitudes of $\Delta\, ABC$ meets the sides of the triangle.

   (c) Find the point H where the three altitudes of $\Delta\, ABC$ meet.

   (d) Find the barycentric coordinates of H relative to the vertices A, B, C.

   (e) Find the tangents of the vertex angles in $\Delta\, ABC$.

2. Do you see how to construe your calculations in the previous exercise as giving a proof that the angle tangents are homogeneous coordinates of the altitude intersection point in any triangle? In other words, is there anything special about the triangle of the previous exercise apart from its location?