Transformation Geometry — Math 331

January 28, 2004

Discussion

- **Terminology Revision.** Any weight 1 linear combination of given points may be called a *barycentric combination* of those points, regardless of whether the coefficients are non-negative.
- **Definition.** A sequence of $r+1$ points $p_0, p_1, \ldots, p_r$ is called *barycentrically independent* if none of them is a barycentric combination of the others.
- **Examples.**
  1. Any two distinct points P, Q are barycentrically independent. If $P \neq Q$, the set of barycentric combinations of P and Q is the line through P and Q.
  2. Three points A, B, C are barycentrically independent if and only if none lies on the line determined by the other two. Thus, the vertices of a triangle are barycentrically independent.
  3. In $\mathbf{R}^3$ the four vertices of a tetrahedron are barycentrically independent.
- **Proposition.** A sequence of $r+1$ points $p_0, \ldots, p_r$ is barycentrically independent if and only for given $a_0, \ldots, a_r$ and given $b_0, \ldots, b_r$ with $a_0 + \ldots + a_r = 1$ and $b_0 + \ldots + b_r = 1$ the following statement is true:

  $$a_0 p_0 + \ldots + a_r p_r = b_0 p_0 + \ldots + b_r p_r \text{ if and only if } a_0 = b_0, \ldots, a_r = b_r \quad .$$
- *Proof.* Obtain this from corresponding facts about linear independence.
- **Theorem.** If $p_0, p_1, \ldots, p_n$ are barycentrically independent points of n-dimensional Euclidean space $\mathbf{R}^n$, and $q_0, q_1, \ldots, q_n$ are any points of $\mathbf{R}^m$, then there is one and only one affine map f from $\mathbf{R}^n$ to $\mathbf{R}^m$ for which $f(p_0) = q_0, f(p_1) = q_1, \ldots, f(p_n) = q_n$.

  Proof. Use the fact that there is a unique linear map taking prescribed values at the members of a basis of $\mathbf{R}^n$.
- **Theorem** If a map f from $\mathbf{R}^n$ to $\mathbf{R}^m$ preserves barycentric combinations, then it must be an affine map.

  Proof. Use two facts: (1) an affine map that carries 0 to 0 must be linear, and (2) a linear map is always given by a matrix.

Exercises due Friday, January 30

1. Let A, B, C, and D be four points in the plane $\mathbf{R}^2$. Show that the polygonal path (sequence of line segments) from A to B, from there to C, then to D, and back to A is a parallelogram if and only if $A - B + C - D = 0$.
2. Show that an affine transformation of the plane carries a parallelogram to a parallelogram.
3. Show that there is one and only one affine transformation of the plane carrying a given parallelogram to another given parallelogram in a given vertex-matching way.
4. Show that any affine transformation of the plane carries the point where the diagonals of a given parallelogram meet to the point where the diagonals of the image parallelogram meet.
5. Explain why an affine transformation of the 3-dimensional space $\mathbf{R}^3$ must always carry a tetrahedron to a tetrahedron.