Math 331 — Transformation Geometry
Follow-Up Assignment

May 21, 2002

Responses to Short Answer Questions

1. How many affine transformations of the plane $\mathbf{R}^2$ fix (without permutation) the three vertices of a given (non-degenerate) triangle?

   Response: One. A basic principle: there is one and only affine mapping with domain $\mathbf{R}^n$ taking prescribed values at the $n+1$ points in a given affine basis of $\mathbf{R}^n$. The 3 vertices of any triangle form an affine basis of $\mathbf{R}^2$.

2. What type of isometry of the Euclidean plane results from composing the reflections in two different parallel lines?

   Response: A translation. The translation is by a vector whose length is twice the distance between the parallel lines pointing in the direction perpendicular to those lines from the line of the first reflection toward the line of the second.

3. What type of isometry of the Euclidean plane results from composing the reflections in two different parallel lines with a third reflection in a line perpendicular to the first two?

   Response: A glide reflection. By definition a glide reflection is the isometry obtained by following the reflection in a line with the translation by a vector parallel to the line. (The same glide reflection is obtained if the translation precedes the reflection.)

4. Describe the transformation of the Euclidean plane that results when a rotation is followed by a non-identity translation.

   Response: A rotation. The isometry must be orientation-preserving because it is the composition of two orientation-preserving isometries. Its linear part must be the composition of the two linear parts and, therefore, must be equal to the linear part of the given rotation since the linear part of a translation is the identity. So the composition cannot be a translation unless the original rotation is the identity.

5. What condition is needed on an affine transformation of the plane in order that the area of every square be preserved under the transformation?

   Response: Its determinant, i.e., the determinant of its linear part, must be ± 1. The transformation need not be an isometry although every isometry of the plane is indeed area-preserving.

6. What type of isometry of the Euclidean plane *cannot* be obtained by composing a number of reflections smaller than 3?

   Response: A glide reflection. Aside from the class of reflections, there are two other classes of isometries, translations and rotations, in which each example may be obtained as the composition of two reflections.

7. What *specific* type of isometry of the plane has the properties that (a) it has a single fixed point, (b) it is equal to its own inverse isometry, and (c) it is not the identity.

   Response: A rotation by π (or *half turn*). Only rotations and reflections have any fixed points. Each reflection fixes its axis, which is a line. Each rotation has a single fixed point, its center. In order for a rotation to be its own inverse, its angle must either be 0 or π up to integer multiples of 2π.

8. How many transformations of the Euclidean plane are in the group of all isometries that permute the vertices of a given equilateral triangle?

   Response: 6. Three rotations (including the identity) and three reflections.

9. How many isometries are in the smallest group of isometries of the Euclidean plane that contains the two reflections in a given pair of perpendicular lines?

   Response: 4. In addition to the two given reflections the group must contain the square (relative to composition) of each, i.e., the identity, and the product, i.e., the composition of the two reflections, which is the rotation by π about the point where the two lines intersect.

10. What type of isometry of $\mathbf{R}^3$ has a unique fixed point?

Response: A reflective rotation. A reflective rotation is by definition the result of following the rotation about a line with the reflection in a plane that is perpendicular to that line. The unique fixed point of a reflective rotation is the point where its axis (of rotation) intersects its plane (of reflection). The only other classes of isometries of $\mathbf{R}^3$ that have fixed points are rotations, each of which has a line of fixed points, and (mirror) reflections, each of which has a plane of fixed points.

Solved Exercises

1. If A, B, and C are barycentrically independent points and $D = 2B + C - 2A$, express the point where the line AD meets the line BC as a barycentric combination of A, B, and C.

Response: The point is $\frac{2}{3}B + \frac{1}{3}C$.

Since the required point must be on the line BC, its representation as a barycentric combination of A, B, and C must, in fact, involve only B and C. Thus, one is looking for a scalar s such that the point of intersection is $(1-s)B + sC$. Since this point is also on the line AD, there must also be a scalar t such that

$$(1-s)B + sC \ = \ (1-t)A + tD \quad .$$

Using the formula for D one has:

$$\begin{aligned}(1-s)B + sC &= (1-t)A + t(-2A + 2B + C) \\ &= (1-3t)A + 2tB + tC\end{aligned}$$

Since A, B, and C are barycentrically independent, barycentric combinations of them must have the same coefficients. Therefore, $1 - 3t = 0$, $1 - s = 2t$, and $s = t$. From these (redundant) equations $s = t = 1/3$.

2. What specific type of affine transformation of the plane has the *affine* matrix

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ -2 & -2 & 1 \end{pmatrix}$$

relative to an affine basis of the plane?

Response: A dilatation with scaling factor 3 centered at the third member of the affine basis.

This is a question about *affine* geometry. Therefore, one may choose rectangular coordinates so that the given affine basis is

$$((1,0),(0,1),(0,0)) \quad .$$

Relative to this rectangular coordinate system the affine transformation, which fixes the origin, is *linear*. Its matrix when regarded as a linear transformation of $\mathbf{R}^2$ is

$$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \quad .$$

3. Assume that triangle ABC has area 12. Let P be the point $P = \frac{1}{6}(A + 2B + 3C)$. What is the area of ΔCAP? Be sure to explain your reasoning.

Response: 4.

An important theorem states that for any point P inside triangle ABC the subtriangle formed with P and two of the vertices of triangle ABC is the fraction of the area of triangle ABC given by the barycentric coordinate of P relative to the third vertex. In this case the third vertex is B, and, therefore, the area of triangle CAP is $\frac{1}{3}$ of the area of ΔABC.

4. Describe, relative to the classification of isometries of the Euclidean plane, the isometry of the plane that results when the reflection in the first axis for rectangular coordinates is followed by a rotation through the angle $\pi/4$ (counter-clockwise) about the point $(0,1)$.

Response: A glide reflection. More specifically, the glide reflection obtained by following reflection in the line through the origin forming angle $\frac{\pi}{8}$ with the positive first coordinate axis followed by translation in the first quadrant direction parallel to that axis through the length $\sqrt{2-\sqrt{2}}$.

The major point here is that the composition of a reflection with a rotation centered off the axis of the reflection is a glide reflection.

For the specific computation one may represent the given rotation about $(0,1)$ as the composition $\lambda\eta$, in which the reflection η in the vertical coordinate axis is followed by the reflection λ in the line through $(0,1)$ that forms the angle $\frac{\pi}{8}$ (counter-clockwise) with the vertical axis. If ξ denotes reflection in the horizontal axis, then the required isometry, call it f, is $(\lambda\eta)\xi = \lambda(\eta\xi)$. The half turn $\eta\xi$ about the origin may be represented as the composition of the reflections in *any* pair of perpendicular lines through the origin, in particular as the composition $\mu\sigma$ where the axes of σ and μ, respectively, are the lines obtained by a $\frac{\pi}{8}$ rotation of the x-axis and the y-axis, respectively. Then $f = \lambda(\mu\sigma) = (\lambda\mu)\sigma$. Since the axes of λ and μ are parallel, the composition $\lambda\mu$ is a translation, namely the translation described in the answer, while σ is the reflection indicated in the answer.

5. For an arbitrary triangle ABC (with vertices listed in counter-clockwise order) what can be said, in general, about the isometry $\rho_A(\alpha)\circ\rho_B(\beta)\circ\rho_C(\gamma)$, where $\rho_P(\theta)$ denotes the rotation counterclockwise by angle θ about the point P and where α, β, γ are, respectively, the vertex angles at A, B, C, respectively?

   Response: Rotation by the angle π (half turn) about the point where the circle inscribed in the triangle intersects the side AC.

   Because composition of isometries involves composition of their linear parts, because the linear parts of rotations compose simply by adding the angles of rotation, and because the sum of the vertex angles of any triangle is π, it is clear that the required isometry is a half turn. The only issue then is what is its center.

   If l is a line, let σ_l denote the reflection in l. For the given triangle let a, b, c, respectively, denote the lines opposite the vertices A, B, C, respectively, and let d, e, f, respectively, denote the corresponding angle bisecting lines. Let M be the point in triangle ABC where d, e, f meet.

   One has

$$\rho_A(\alpha) = \sigma_d\sigma_c = \sigma_b\sigma_d \quad \rho_B(\beta) = \sigma_e\sigma_a = \sigma_c\sigma_e \quad \rho_C(\gamma) = \sigma_f\sigma_b = \sigma_a\sigma_f \ .$$

   Thus

$$\rho_A(\alpha)\rho_B(\beta) = (\sigma_d\sigma_c)(\sigma_c\sigma_e) = \sigma_d(\sigma_c\sigma_c)\sigma_e = \sigma_d\sigma_e \ ,$$

   which is the rotation $\rho_M(\alpha+\beta)$. Recalling that the center of the inscribed circle is the point M and that each of the three line segments drawn from M perpendicularly to a side of the triangle is a radius of the inscribed circle, one notes that the point Q of the inscribed circle on the side AC is the foot of the perpendicular h drawn from M to AC. . The angle from f to h has measure $\frac{\pi}{2} - \frac{\gamma}{2} = \frac{\alpha}{2} + \frac{\beta}{2}$. Therefore,

$$\rho_A(\alpha)\rho_B(\beta) = \sigma_d\sigma_e = \rho_M(\alpha+\beta) = \sigma_h\sigma_f \quad .$$

   Hence,

$$\begin{aligned}\rho_A(\alpha)\rho_B(\beta)\rho_C(\gamma) &= (\sigma_h\sigma_f)(\sigma_f\sigma_b)\\ &= \sigma_h(\sigma_f\sigma_f)\sigma_b\\ &= \sigma_h\sigma_b\\ &= \rho_Q(\pi)\end{aligned}$$