Math 220 Review Slides
on Inner Products

http://math.albany.edu/pers/hammond/course/mat220/
Course   Assignments   Slides

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1 Inner Products

The notion of *inner product*

1. generalizes the "dot product" in $\mathbf{R}^n$
2. is coordinate-free
3. makes it possible in abstract contexts to speak of
   (a) lengths
   (b) angles

2 Abstract Inner Products

Definition. An *inner product* on a vector space V is a function I of two variables from V that takes scalar values and satisfies the following rules:

1. $I(c_1v_1 + c_2v_2, v_3) = c_1I(v_1, v_3) + c_2I(v_2, v_3)$
2. $I(v_3, c_1v_1 + c_2v_2) = c_1I(v_3, v_1) + c_2I(v_3, v_2)$
3. $I(v_1, v_2) = I(v_2, v_1)$
4. $I(v, v) > 0$ provided $v \neq 0$

3 Inner Products: Example 1

Ordinary "Dot" Product

$$V = \mathbf{R}^n \qquad I(v, w) = v \cdot w = v_1w_1 + v_2w_2 + \ldots + v_nw_n$$

4 Inner Products: Example 2

Inner Product given by a
Postiive-Definite Symmetric Matrix

$$\begin{aligned} V &= \mathbf{R}^2 \\ S &= \begin{pmatrix} a & b \\ b & c \end{pmatrix} \text{ where } ac - b^2 > 0 \text{ and } a + c > 0 \\ I(v, w) &= {}^t vSw = av_1w_1 + bv_1w_2 + bv_2w_1 + cv_2w_2 \end{aligned}$$

5 Inner Products: Example 3

$$V \quad = \quad \mathcal{P}_d \quad = \quad \{\text{polynomials of degree } \le d\}$$

A inner product on V for each interval $a \le t \le b$ ($a < b$):

$$I(f, g) \;=\; \int_a^b f(t)g(t)\,dt$$

6 Cauchy-Schwarz Inequality

Theorem. *If I is an inner product on V, then for all v, w in V*

$$|I(v, w)| \le \sqrt{I(v, v)}\sqrt{I(w, w)} \quad .$$

Moreover, when $v \ne 0$, equality occurs if and only if there is a scalar c such that $w = cv$.

7 Length of a vector

Relative to an inner product I:

$$\text{length of } v \quad = \quad \|v\|_I \quad = \quad \sqrt{I(v, v)}$$

8 Distance between two points

Relative to an inner product I:

$$\text{distance from } P \text{ to } Q \quad = \quad \|Q - P\|_I$$

9 Angle between two vectors

Relative to an inner product I, when $v, w \ne 0$:

$$\angle_I(v, w) \quad = \quad \arccos\left(\frac{I(v, w)}{\|v\|_I\,\|w\|_I}\right)$$

10 Orthogonality

Perpendicularity (or orthogonality) relative to an inner product I

$$v \perp w \text{ if and only if } I(v, w) \;=\; 0$$

11 Parallelism

Relative to an inner product I

$$v \parallel w \text{ if and only if } |I(v, w)| \;=\; \|v\|_I\,\|w\|_I$$

12 Orthonormal bases

Definition. A basis $\mathbf{v} = (v_1 v_2 \ldots v_n)$ of an n-dimensional vector space with an inner product I is an *orthonormal basis* relative to I if $v_1, v_2, \ldots, v_n$ are mutually perpendicular vectors of length 1 (relative to I).

Equivalently, relative to I,

$$\mathbf{v} \text{ is an orthonormal basis if and only if } I(v_j, v_k) = \begin{cases} 0 & \text{if} j \neq k \\ 1 & \text{if} j = k \end{cases}$$

13 Orthogonal matrices

Let U be an $n \times n$ matrix. The following conditons on U are equivalent:

1. U is an orthogonal matrix.
2. U is invertible and $U^{-1} = {}^tU$.
3. The n columns of U form an orthonormal basis of $\mathbf{R}^n$ relative to the standard inner product (the "dot" product).
4. The n rows of U form an orthonormal basis of $\mathbf{R}^n$ relative to the standard inner product.

14 Orthogonal linear maps

Definition. If V is a vector space with an inner product I and $V \xrightarrow{\varphi} V$ a linear map, φ is said to be an *orthogonal linear map* relative to I if φ is invertible and if one has

$$I(\varphi(v), \varphi(w)) = I(v, w) \text{ for all } v, w \text{ in } V \quad .$$

Note: If V is finite-dimensional, it is redundant to require that φ should be invertible when φ is required to preserve the inner product.

15 Preservation of Distances

Theorem. *If V is a vector space with an inner product I and $V \xrightarrow{\varphi} V$ a linear map, then φ is an orthogonal linear map if and only if φ is invertible and length-preserving, i.e., for each v in V one has $\|\varphi(v)\| = \|v\|$.*

Note: If V is finite-dimensional, it is redundant to require that φ should be invertible when φ is required to preserve lengths.

16 Orthogonal Linear Maps and Orthogonal Matrices

Theorem. *If V is an n-dimensional vector space, I an inner product on V, $V \xrightarrow{\varphi} V$ a linear map, $\boldsymbol{v} = (v_1 v_2 \ldots v_n)$ an orthonormal basis of V relative to I, and M the matrix of φ relative to $\boldsymbol{v}$, then φ is an orthogonal linear map relative to I if and only if M is an orthogonal matrix.*