Math 220 Class Slides

http://math.albany.edu/pers/hammond/course/mat220/
Course Assignments Slides

May 1, 2008

1 Matrix of a Linear Map for a Pair of Bases

- 3 ways to characterize the matrix of

$$V \xrightarrow{\phi} W$$

relative to bases

$$\mathbf{v} \text{ and } \mathbf{w}$$

1. The transport diagram:

$$\begin{array}{ccc} V & \xrightarrow{\phi} & W \\ \alpha_{\mathbf{v}} \uparrow & & \uparrow \alpha_{\mathbf{w}} \\ \mathbf{R}^n & \xrightarrow[f_M]{} & \mathbf{R}^m \end{array}$$

where

$$f(x) = Mx$$

2. $$M_{ij} = i\text{-th coordinate of } f(v_j) \text{ relative to } \mathbf{w}$$

3. $$f(\mathbf{v}) = \mathbf{w}M$$

2 April 29, Exercise No. 2

Problem: Find the matrix of the reflection of $\mathbf{R}^3$ in the plane

$$6x - 2y + 3z = 0$$

Solution:

- Let

$$\sigma = \text{the reflection} \qquad \pi = \text{projection of } \mathbf{R}^3 \text{ on the plane}$$

- $$\sigma(p) - \pi(p) = \pi(p) - p$$

- $$\sigma = 2\pi - 1$$

- $$\lambda = \text{projection of } \mathbf{R}^3 \text{ on the normal vector } N$$
- $$\pi = 1 - \lambda$$
- $$\sigma = 1 - 2\lambda$$
- $$\lambda(p) = \left(\frac{p \cdot N}{N \cdot N}\right) N \quad N = (6, -2, 3)$$
- $$\lambda((x,y,z)) = L \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad L = \frac{1}{49} \begin{pmatrix} 36 & -12 & 18 \\ -12 & 4 & -6 \\ 18 & -6 & 9 \end{pmatrix}$$
- $$\text{required matrix} = 1 - 2L = \frac{1}{49} \begin{pmatrix} -23 & 24 & -36 \\ 24 & 41 & 12 \\ -36 & 12 & 31 \end{pmatrix}$$

3 April 29, Exercise No. 4

Problem: Let S be the 3×3 matrix

$$\begin{pmatrix} 10 & -6 & -2 \\ -6 & 5 & -8 \\ -2 & -8 & 3 \end{pmatrix} .$$

Find an orthogonal matrix U and a diagonal matrix D such that

$$S = UDU^{-1} .$$

Solution:

- Characteristic polynomial: $X(t) = t^3 - 18t^2 - 9t + 810$
- Eigenvalues — roots of $X(t)$: -6, 9, 15
- Eigenvectors (in order):

$$\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

- $$D = \begin{pmatrix} -6 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 15 \end{pmatrix} \qquad U = \frac{1}{3} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

4 May 1, Exercise No. 2

Problem: Give an example of a 2×2 matrix having eigevalues 1 and -1 where the corresponding eigenvectors form the angle $\pi/4$.

Solution:

- Begin with the diagonal matrix having 1 and -1 as eigenvalues.

- $$D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
- $(1,0)$ and $(0,1)$ are the eigenvectors of D.
- Perform a change of basis
$$\mathbf{v} = \mathbf{e}Q$$
where the columns of Q form the angle $\pi/4$.
- $$Q = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$
- $$\text{required matrix } M = QDQ^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix}$$
- **Note:**
  1. Recall for any matrices G and H one has
  $$(GH)_j = GH_j ,$$
  where j as a subscript on a matrix indicates the j-th column.
  2. The j-th column of a diagonal matrix is a scalar times the j-th column of the identity matrix.
  3. Apply these observations to the relation
  $$MQ = QD$$
  to understand why the columns of Q are eigenvectors of M.

5 May 1, Exercise No. 3

Problem: Show that the matrix
$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$
is not similar to a diagonal matrix.

Solution:

- Characteristic polynomial: $t^2 - 4t + 4 = (t-2)^2$
- Only one eigenvalue: 2
- Eigenspace for the eigenvalue 2 has dimension 1
- No basis consisting of eigenvectors
- Hence, not diagonalizable.

6 May 1, Exercise No. 4

Problem: Let S be the 3×3 symmetric matrix
$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix} .$$

1. Find an orthogonal matrix U and a diagonal matrix D such that
$$U^{-1}SU = D .$$

2. What is the largest value achieved on the unit sphere $x_1^2 + x_2^2 + x_3^2 = 1$ by the function

$$h(x) = {}^t x S x = 2x_1^2 + 3x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3 \text{ ?}$$

Solution:

- Characteristic polynomial: $X(t) = t^3 - 7t^2 + 14t - 8$
- Eigenvalues — roots of $X(t)$: 1, 2, 4
- Eigenvectors (in order):

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

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$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \qquad U = \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix}$$

- Since U is an orthogonal matrix, Uv is on the unit sphere if and only if v is on the unit sphere.
- Write x = Uv and compute $h(x)$ as a function of v.
- Since U is an orthogonal matrix, $U^{-1} = {}^tU$.
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$$h(x) = h(Uv) = {}^t(Uv)S(Uv) = {}^tv\,{}^tUSUv = {}^tvDv = v_1^2 + 2v_2^2 + 4v_3^2$$

- The maximum value of $h(x) = h(Uv)$ when $\|v\| = \sqrt{v \cdot v} = 1$ is 4.