Math 220 Class Slides

http://math.albany.edu/pers/hammond/course/mat220/
Course Assignments Slides

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1 Parametric Representations, Coordinates, and Bases

Recall:

- To give a parametric representation of a linear subspace in a vector space is to represent a general member of the subspace as a linear combination of the vectors in some basis of the subspace.
- The coefficients of the basis in such a representation are "coordinates" in the linear subspace relative to the basis.
- To have coordinates for the points of a linear subspace of dimension k is to have a linear way of matching points in the subspace with points in $\mathbf{R}^k$.
- To have coordinates for the points of a linear subspace of dimension k is to have an isomorphism from $\mathbf{R}^k$ to the subspace.

2 Isomorphisms

Definition. Let V, W be vector spaces. An *isomorphism* from V to W is a linear map $V \xrightarrow{\phi} W$ that establishes a one-to-one correspondence of elements of V with elements of W.

Proposition. If there is an isomorphism from V to W, then there is an "inverse" isomorphism from W to V.

Proof. The inverse of ϕ is an isomorphism from W to V.

Definition. V and W are isomorphic vector spaces if there is an isomorphism from one to the other.

If U is isomorphic with V and V is isomorphic with W, then U is isomorphic with W.

Proof. Compose an isomorphism from U to V with an isomorphism from V to W.

3 Isomorphisms and Dimension

Theorem. If V and W are isomorphic vector spaces, then $\dim V = \dim W$.

Proof. It is an exercise to show that if $v_1, v_2, \ldots, v_n$ is a basis of V, then $\phi(v_1), \phi(v_2), \ldots, \phi(v_n)$ is a basis of W.

Theorem. Any vector space of dimension n is isomorphic to $\mathbf{R}^n$.

Proof. If $\mathbf{v} = v_1, v_2, \ldots, v_n$ is a basis of V, then the linear map

$$\mathbf{R}^n \xrightarrow{\alpha_{\mathbf{v}}} V$$

that is defined by

$$\alpha_{\mathbf{v}}(x) \;=\; (v_1 v_2 \ldots v_n) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \;=\; x_1 v_1 + x_2 v_2 + \ldots x_n v_n$$

is an isomorphism from $\mathbf{R}^n$ to V.

4 Coordinates with respect to a Basis

- **Note:** When $v_1, \ldots, v_r$ are linearly independent, the coefficients for a given linear combination of them are unique:

$$x_1 v_1 + \ldots + x_r v_r \;=\; y_1 v_1 + \ldots + y_r v_r \;\text{ if and only if }\; x_1 \;=\; y_1,\; \ldots,\; x_r \;=\; y_r \quad .$$

- **Definition.** If $v_1, \ldots, v_n$ form a basis of V, then $x_1, \ldots, x_n$ are called the *coordinates of v with respect to $v_1, \ldots, v_r$* when

$$v \;=\; x_1 v_1 + \ldots + x_n v_n \quad .$$

- **Example:** 2, -1, and 3 are the coordinates of the point $(2, -1, 3)$ with respect to the standard basis of $\mathbf{R}^3$.
- **Example:** 2, -1, and 3 are the coordinates of the polynomial $3t^2 - t + 2$ with respect to the basis $\{1, t, t^2\}$ of the 3-dimensional vector space P_2 consisting of all polynomials with degree at most 2 in the variable t.
- The order in which the members of a basis are listed affects the ordering of coordinates taken with respect to that basis.

5 Linearity in the Euclidean Case

Recall:

Theorem. For any linear map $\mathbf{R}^n \xrightarrow{\phi} \mathbf{R}^m$ between Euclidean spaces there is a unique $m \times n$ matrix M such that $\phi = f_M$.

Re-stated: Every linear map from $\mathbf{R}^n$ to $\mathbf{R}^m$ is given in the usual way by some matrix.

6 The Fundamental Theorem on Linear Maps

Theorem. If $V \xrightarrow{\phi} W$ is a linear map between vector spaces with V finite-dimensional, then

$$\dim(V) \;=\; \dim(\text{Kernel}\phi) + \dim(\text{Image}\phi)$$

Proof when both V and W are finite-dimensional:
Let

$$n \;=\; \dim V \;\text{ and }\; m \;=\; \dim W \quad .$$

Let

$$\mathbf{v} \;=\; (v_1 v_2 \ldots v_n) \;\text{ and }\; \mathbf{w} \;=\; (w_1 w_2 \ldots w_m)$$

be bases of V and W.

Use $\alpha_{\mathbf{v}}$ and $\alpha_{\mathbf{w}}$ to "transport" ϕ to

$$\mathbf{R}^n \xrightarrow{f} \mathbf{R}^m \quad .$$

The transport of ϕ is the linear map f in this diagram:

$$\begin{array}{ccc} V & \xrightarrow{\phi} & W \\ \alpha_{\mathbf{v}} \uparrow & & \uparrow \alpha_{\mathbf{w}} \\ \mathbf{R}^n & \xrightarrow[f]{} & \mathbf{R}^m \end{array} \quad .$$

f is defined by

$$f \;=\; \alpha_{\mathbf{w}}^{-1} \circ \phi \circ \alpha_{\mathbf{v}} \quad .$$

Since $\alpha_{\mathbf{v}}$ and $\alpha_{\mathbf{w}}$ are isomorphisms, one has

$$\mathrm{dimKer}(\phi) \;=\; \mathrm{dimKer}(f) \;\text{ and }\; \mathrm{dimIm}(\phi) \;=\; \mathrm{dimIm}(f) \quad .$$

So the theorem is proved by "transport" to the Euclidean case.

7 Matrix of a Linear Map for a Pair of Bases

The transport diagram:

$$\begin{array}{ccc} V & \xrightarrow{\phi} & W \\ \alpha_{\mathbf{v}} \uparrow & & \uparrow \alpha_{\mathbf{w}} \\ \mathbf{R}^n & \xrightarrow[f]{} & \mathbf{R}^m \end{array}$$

The linear map f between Euclidean spaces has a matrix M

$$f(x) \;=\; f_M(x) \;=\; Mx$$

Definition. M is called the *matrix of ϕ for the pair of bases*

$$\mathbf{v} \;=\; (v_1 v_2 \ldots v_n) \;\text{ and }\; \mathbf{w} \;=\; (w_1 w_2 \ldots w_m) \quad .$$

8 Exercise No. 1

- **Task:** If possible, invert the 4×4 matrix

$$M \;=\; \begin{pmatrix} 1 & 2 & 1 & 2 \\ -2 & -1 & 3 & 2 \\ -2 & 2 & 6 & -1 \\ 1 & 0 & -2 & 0 \end{pmatrix}$$

- Form the 4×8 matrix

$$\begin{pmatrix} M & 1_4 \end{pmatrix}$$

that augments M with the 4×4 identity matrix 1_4, and use row operations to maneuver the first 4 columns of that into reduced row echelon form.

- In this case the RREF of the first 4 columns is 1_4 so the last 4 columns of the reduced matrix form the inverse of M, which is:

$$M^{-1} = \begin{pmatrix} 2 & -4 & -4 & -17 \\ -1 & 7/3 & 8/3 & 11 \\ 1 & -2 & -2 & -9 \\ 0 & 2/3 & 1/3 & 2 \end{pmatrix} \quad .$$

9 Exercise No. 2(b)

- **Task:** For the following 4×4 matrix M find

  (a) the rank of the matrix
  (b) a non-redundant set of linear equations in 4 variables that characterizes the linear relations among the rows of the matrix.

- **Note:** As explained in the previous class, this is essentially the same problem as that of finding linear equations for the image of the linear map

$$f_M(x) = Mx \quad .$$

- The matrix:

$$\begin{pmatrix} 1 & 2 & -4 & 7 \\ -2 & -1 & -1 & -8 \\ 5 & 7 & -11 & 29 \\ -3 & -6 & 12 & -21 \end{pmatrix}$$

- The RREF of its **transpose**:

$$\begin{pmatrix} 1 & 0 & 3 & -3 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- The rank of M is 2.

- A non-redundant characterizing set of row relations:

$$\begin{cases} -3y_1 + y_2 + y_3 = 0 \\ 3y_1 + y_4 = 0 \end{cases}$$