Linear Algebra Handout: Two Solved Examples

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1. Let W_1 and W_2 be the subspaces of $\mathbf{R}^3$ given by

$$W_1 \ = \ \text{span}\,\{(1,2,3),(2,1,1)\} \quad \text{and} \quad W_2 \ = \ \text{span}\,\{(1,0,1),(3,0,-1)\} \quad .$$

Find a set of generating vectors for $W_1 \cap W_2$.

Response. W_1 and W_2 are each spanned by two linearly independent vectors, and for that reason each is a plane through the origin in $\mathbf{R}^3$. One therefore expects their intersection to be a line through the origin in $\mathbf{R}^3$, i.e., the set of all scalar multiples of a single vector. The strategy adopted is to find equations defining W_1 and W_2 and then solve both equations simultaneously to find $W_1 \cap W_2$.

For W_1 one seeks a vector (a,b,c) such that

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad .$$

Solving the corresponding linear system of equations using row operations, one finds that the most general such vector (a,b,c) is a scalar multiple of $(1,-5,3)$, and "the" equation of W_1 is $x - 5y + 3z \ = \ 0$.

A similar method can be used for W_2. On the other hand, one sees quickly by inspection that "the" equation of W_2 is $y \ = \ 0$. Then solving the two equations simultaneously — for example, using row operations on the matrix

$$\begin{pmatrix} 1 & -5 & 3 \\ 0 & 1 & 0 \end{pmatrix}$$

— one finds that a basis of $W_1 \cap W_2$ is given by the single vector $(-3,0,1)$, i.e.,

$$W_1 \cap W_2 \ = \ \text{span}\,\{(-3,0,1)\} \quad .$$

2. Find a particular solution of the linear differential equation

$$y'' + 4y \ = \ x^2 \quad .$$

Response. In the study of differential equations one learns to look for particular solutions in various ways. In this case it will be fruitful to propose

$$y \ = \ a + bx + cx^2$$

as a trial solution. With this y one finds $y'' \ = \ 2c$, and, therefore,

$$y'' + 4y \ = \ (4a + 2c) + 4bx + 4cx^2 \quad .$$

To have the right-hand side evaluate as x^2 one needs

$$\begin{aligned} 4a + 2c &= 0 \\ 4b &= 0 \\ 4c &= 1 \end{aligned}$$

Solving this system for a, b, c one obtains

$$y \ = \ \frac{1}{4}x^2 - \frac{1}{8} \quad .$$