Summary Review for Math 106

Basic

1. Slope of a line: from any point to another, the change in y divided by the change in x
2. General form of the equation of a line: $ax + by = c$
3. Equation of the line through (a, b) with slope m:
$$\frac{y-b}{x-a} = m$$
4. Equation of the line through (a, b) and (c, d)
$$\frac{y-b}{x-a} = \frac{d-b}{c-a}$$
5. A curve is a graph when it meets vertical lines once
6. The slope of a curve at a point is the slope of the line tangent to the curve at the given point
7. The slope of the graph of f at a point is the value of the derivative f' at the first coordinate of the given point
8. $f'(x) =$ slope of tangent to graph of f at $(x, f(x))$
9. Definition of the derivative as limit of the "difference quotient":
$$f'(x) = \lim_{t \to 0} \frac{f(x+t) - f(x)}{t}$$
10. $f'' =$ derivative of $f' =$ the second derivative of f

Formulas for Derivatives

1. If $f = c =$ constant, then $f' = 0$
2. $(f + g)' = f' + g'$
3. $(f - g)' = f' - g'$
4. $(c_1 f_1 + c_2 f_2 + \ldots + c_n f_n)' = c_1 f_1' + c_2 f_2' + \ldots c_n f_n'$
5. The product rule:
$$(fg)' = fg' + gf'$$
6. The power rule: If $f(x) = x^a$, then $f'(x) = ax^{a-1}$.
7. The quotient rule:
$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$
8. Composition of two functions:
$$(f \circ g)(x) = f(g(x)) \qquad (\text{“}f \text{ following } g\text{”})$$
9. The chain rule (for the derivative of a composition):
   (a) Leibniz notation:
$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$
   (b) Functional notation:
$$(f \circ g)' = (f' \circ g) \cdot g'$$

(c) Reconciliation:

$$y = f(x) \quad x = g(t) \quad \frac{dy}{dx} = f'(x) = f'(g(t)) \quad \frac{dx}{dt} = g'(t)$$

10. Generalized power rule (application of chain rule with $f(u) = u^a$):

$$\frac{d}{dx}g(x)^a = ag(x)^{a-1}g'(x)$$

11. The exponential rule: If $f(x) = a^x$, then $f'(x) = L(a)a^x$ where

$$L(a) = \lim_{t \to 0} \frac{a^t - 1}{t}$$

(Note: in this it is assumed that the constant base a is positive.)

Exponentials and Logarithms

1. e $(2 < e < 3)$ is the unique number for which $L(e) = 1$, where L is the multiplier appearing in the exponential rule
2. (Important special case of the exponential rule)

$$\frac{d}{dx}e^x = e^x$$

3. Secondary school definition of logarithm:

$$c = \log_a(b) \text{ exactly when } a^c = b \quad (a, b > 0)$$

4. L spawns all logarithms:

$$\log_a(b) = \frac{L(b)}{L(a)} \quad (a, b > 0)$$

5. L is logarithm for the base e or the "natural logarithm":

$$L(a) = \log_e(a) \text{ for each } a > 0$$

6. Derivative of L:

$$L'(x) = \frac{1}{x} \quad (x > 0)$$

7. Derivative of $\log_a$:

$$\frac{d}{dx}\log_a(x) = \frac{1}{L(a)x} \quad (a, x > 0)$$

Graph Sketching

1. Qualitatively accurate sketches may be obtained by plotting only a few points and taking account of information about
   (a) where the function is increasing and decreasing
   (b) where the function is concave up and concave down
   (c) points where the function has local extremes
   (d) points of inflection
   (e) horizontal and vertical asymptotes
2. f is increasing where $f' > 0$, decreasing where $f' < 0$
3. f is concave up where $f'' > 0$, concave down where $f'' < 0$
4. $f'(c) = 0$ if f has a local maximum or minimum when $x = c$
5. $f''(c) = 0$ if the graph of f has an inflection point when $x = c$
6. the line $y = b$ is a horizontal asymptote if $f(x) \to b$ as $x \to \infty$ or as $x \to -\infty$
7. the line $x = a$ is a vertical asymptote if $f(x)$ becomes infinite (positively or negatively) as $x \to a$

Exponential Growth (or Decay)

1. The model: $A(t) = Ce^{kt}$ where

   $A(t)$ is the amount at time t.
   $C = A(0)$ is the initial amount.
   k is the "growth constant".

2. The differential equation: $A'(t) = kA(t)$.

   Exponential growth (or decay) is characterized by the *relative rate of change*

   $$\frac{A'(t)}{A(t)}$$

   being a constant k. $k > 0$ for growth, while $k < 0$ for decay.

3. Examples.

   Bacterial growth.
   Radioactive decay.
   Money on deposit at a given interest rate continuously compounded.

4. Interest.

   Various forms of compounding at interest rate r per year (percentage rate $100r$) with initial deposit P and $A(t)$ the amount on account after t years.

   Annual compounding: $A(t) = P(1+r)^t$.
   Semi-annual compounding: $A(t) = P(1+r/2)^{2t}$.
   Periodic compounding m times per year: $A(t) = P(1+r/m)^{mt}$.
   Continuous compounding (limiting case as number of periods per year increases):

   $$A(t) = \lim_{m\to\infty} P\left(1+\frac{r}{m}\right)^{mt} = Pe^{rt}$$

5. Doubling time and half life.

   With a given model of exponential growth (or decay), i.e., for a given growth (or decay) constant k, the change ratio

   $$\frac{A(t+u)}{A(t)}$$

   for a time interval u depends only on u and has the value e^{ku}.

   For $k > 0$ the value of u for which the change ratio is 2 is called the *doubling time*: $u = \frac{\ln 2}{k}$.
   For $k < 0$ the value of u for which the change ratio is $1/2$ is called the *half life*: $u = -\frac{\ln 2}{k}$.

Integration

There are two kinds of integrals: $\int f(x)\,dx$ is an indefinite integral, while $\int_a^b f(x)\,dx$ is a definite integral. An indefinite integral is a function, while a definite integral is a number. Indefinite integrals provide, via the fundamental theorem of calculus, the principal way of evaluating definite integrals.

1. Indefinite integrals.

   A function F is called an *anti-derivative* of a function f if f is the derivative of F ($F' = f$). The indefinite integral $\int f(x)\,dx$ is understood to denote the most general anti-derivative of f. Any two anti-derivatives of f differ by a constant. For example, $\int 2x\,dx = x^2 + C$, where C is an arbitrary constant, since $\frac{d}{dx}x^2 = 2x$.

2. Definite integrals.

   The definite integral $\int_a^b f(x)\,dx$ of a function f on an interval $a \le x \le b$ is by definition the limit, when it exists, taken over all finite subdivisions of the interval, of the Riemann sums of f for the subdivisions. When $f(x) \ge 0$ for $a \le x \le b$, the definite integral may be interpreted as the *area under the graph of* f, i.e., the area of the region between the graph of f and the horizontal axis for $a \le x \le b$.

3. The area between two graphs.

   When $f(x) \le g(x)$ for $a \le x \le b$, the graph of g lies above the graph of f within the interval, and if A denotes the area between these graphs within the interval, then

$$A = \int_a^b (g(x) - f(x))\, dx \quad .$$

4. The fundamental theorem of calculus.

   Theorem. *If f is continuous for $a \le x \le b$, and if F is an anti-derivative of f, then*

$$\int_a^b f(x)\, dx = F(b) - F(a) \quad .$$

   Notation:

$$[F(x)]_a^b = F(b) - F(a)$$

   Example: The area under the hyperbola $y = \frac{1}{x}$ between the vertical lines $x = 3$ and $x = 9$ is the definite integral of $\frac{1}{x}$ for $3 \le x \le 9$. By the fundamental theorem

$$\int_3^9 \frac{1}{x}\, dx = [\ln x]_3^9 = \ln 9 - \ln 3 = \ln \frac{9}{3} = \ln 3 \quad .$$

5. Rules for finding anti-derivatives.

   These rules arise from reversing rules for differentiation.

   (a)

$$\int f(x) + g(x)\, dx = \int f(x)\, dx + \int g(x)\, dx$$

   (b)

$$\int cf(x)\, dx = c \int f(x)\, dx$$

   (c)

$$\int x^m\, dx = \begin{cases} \frac{x^{m+1}}{m+1} + C & \text{for } m \ne -1 \\ \ln x + C & \text{for } m = -1 \end{cases}$$

   (d)

$$\int e^{kx}\, dx = \frac{e^{kx}}{k} \text{ for } k \ne 0$$

   (e) Substitution rule

$$\int f(g(x))g'(x)\, dx = f(g(x)) + C$$

   (f) Integration by parts

$$\int f(x)g'(x)\, dx = f(x)g(x) - \int g(x)f'(x)\, dx$$

Functions of Several Variables

A function f of N variables $x_1, x_2, \ldots x_N$ has a *partial derivative* with respect to each variable. The partial derivative of f with to the variables x_j (for $j = 1, 2, \ldots, N$) is the derivative of the function $\frac{\partial f}{\partial x_j}$ of one variable obtained by holding all variables other than x_j constant. Of course, this partial derivative depends not only on x_j but also on the temporarily constant values of the other values.

Example. Suppose

$$f(x,y,z) \;=\; x^2 y e^z + y^3 z^2 \quad .$$

Then

$$\begin{aligned} \frac{\partial f}{\partial x} &= 2xye^z \\ \frac{\partial f}{\partial y} &= x^2 e^z + 3y^2 z^2 \\ \frac{\partial f}{\partial z} &= x^2 y e^z + 2y^3 z \end{aligned}$$

Since a partial derivative of f is a function of the same variables as f, one may consider the partial derivatives of a partial derivative. Thus, the partial derivative with respect to x of the partial derivative of f with respect to z is a *second order partial derivative.* There is a notation for second order partial derivatives:

$$\frac{\partial^2 f}{\partial x \partial z} \;=\; \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right) \quad .$$

From the previous example:

$$\frac{\partial^2 f}{\partial z \partial y} \;=\; x^2 e^z + 6y^2 z \;=\; \frac{\partial^2 f}{\partial y \partial z}$$

Under very mild conditions on a function having second order partial derivatives relations of equality like $\frac{\partial}{\partial z}\frac{\partial}{\partial y} = \frac{\partial}{\partial y}\frac{\partial}{\partial z}$ are true.

Extreme Values

With every extreme value problem, i.e., minimum value problem or maximum value problem, the problem involves not only a function but also a set of values for the variables on which the function depends. One can be asked to find either the extreme value in question or the point (or points) in the domain under consideration where the extreme value occurs.

Extreme values of a function of one variable on an interval must occur either at an endpoint or at a point inside the interval where the derivative of the function is zero, i.e., where the graph of the function has a horizontal tangent, or at a point where the graph of the function has no tangent.

Extreme values of a function f of N variables on a domain may occur at points of the domain where all partial derivatives $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_N}$ are zero or at points where one (or more) of these partial derivatives fails to exist. Aside from that extreme values may occur at *boundary points* of the domain. The case of boundary points is analogous to the case of endpoints for a function of one variable. Typically a boundary point is characterized by an equation, called a *constraint*, of the form

$$g(x_1, x_2, \ldots, x_N) \;=\; 0 \quad .$$

While it is possible for more than one constraint equation to be involved, the content of this course is limited to the case of one constraint. For that case the method of *Lagrange multipliers* is based on this:

Theorem. *The extreme values of $f(x_1, x_2, \ldots, x_N)$ subject to the constraint $g(x_1, x_2, \ldots, x_N) = 0$ must occur at points of the constraint set where the partial derivatives of f are "aligned" with the partial derivatives of g in the sense that there is a number t (the Lagrange multiplier) for which the N equations*

$$\frac{\partial f}{\partial x_j} \;=\; t\frac{\partial g}{\partial x_j} \qquad j \;=\; 1, 2, \ldots, N$$

are satisfied.

These equations form a system of N equations in the $N+1$ variables $x_1, \ldots, x_N$ and t. Adding the constraint equation $g(x_1, \ldots, x_N) = 0$ gives a system of $N+1$ equations in $N+1$ variables that, in principle, one can solve.

Example. Find the maximum and minimum values of the function $f(x,y) = 5x^2 + 4xy + 2y^2$ in the disk $x^2 + y^2 \leq 1$.

Solution. One finds

$$\frac{\partial f}{\partial x} = 10x + 4y, \quad \frac{\partial f}{\partial y} = 4x + 4y \quad .$$

There is one point, namely the origin $x = y = 0$ where $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$, and an extreme value is possible there. There are no points where $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ fail to exist. Therefore, any other possible points where extreme values can occur must be boundary points. Boundary points in this example are points on the circle $g(x,y) = x^2 + y^2 - 1 = 0$. One has

$$\frac{\partial g}{\partial x} = 2x, \quad \frac{\partial g}{\partial y} = 2y \quad .$$

The principle of Lagrange multipliers says that other possible extreme values must occur at points where

$$\frac{\partial f}{\partial x} = t\frac{\partial g}{\partial x}, \quad \frac{\partial f}{\partial y} = t\frac{\partial g}{\partial y} \text{ and } g(x,y) = 0 \quad .$$

These equations are:

$$10x + 4y = t \cdot 2x, \quad 4x + 4y = t \cdot 2y, \quad \text{and} \quad x^2 + y^2 = 1 \quad .$$

Eliminating t from the first two of these equations gives

$$10 + 4m = \frac{4}{m} + 4 \text{ where } m = \frac{y}{x} \quad .$$

This simplifies to the quadratic equation $4m^2 + 6m - 4 = 0$ which has roots $m = -2$ and $m = 1/2$. For each of these two values of m there are two points (x,y) satisfying $x^2 + y^2 = 1$, in which $x = \pm 1/\sqrt{1+m^2}$ and $y = mx$. When $m = -2$, $f(x,y) = 1$, and when $m = 1/2$, $f(x,y) = 6$. Thus, the minimum value of f in the disk $x^2 + y^2 \leq 1$ is 0 (taken at the origin), and the maximum is 6 (given by $m = 1/2$). Note also that the minimum value of f on the circle $x^2 + y^2 = 1$ is 1 (given by $m = -2$).