Notes on Newton's Method

Supplementary Material for Honors Calculus
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Proposition: Let f be a function that is differentiable on an interval I and assume further:

(a) The derivative f' is either positive throughout I or negative throughout I.
(b) The derivative f' is either steadily increasing in I or steadily decreasing in I.
(c) There are points a, b in I for which $f(a)$ and $f(b)$ are both non-zero with opposite sign.

Then:

1. There is one and only one point z in I for which $f(z) = 0$.
2. If x is on the **convex** side of the graph of f in I, then so is $x' = x - f(x)/f'(x)$, and x' lies between x and z.
3. Successive iterations of Newton's method beginning with a point x on the **convex** side of the graph of f in I will converge to z.
4. *Error control principle.* If c is any point in I on the **concave** side of the graph of f and x is on the **convex** side, then the distance between x and z is at most the absolute value of $f(x)/f'(c)$.

Proof: If f' is positive in I one has $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I. If instead f' is negative in I, then one has $f(x_1) > f(x_2)$ for $x_1 < x_2$. For this reason there is *at most* one root z in I with $f(z) = 0$. The *Intermediate Value Theorem for Continuous Functions* guarantees that there is at least one root between a and b.

We shall assume that f' is positive and increasing. One may reduce each of the other three cases to this case by reflecting either in the horizontal axis or in the vertical line $x = z$ or both. Under this assumption the convex side of the graph of f is the right side. Suppose that $z < x$: then $0 = f(z) < f(x)$. We apply the Mean Value Theorem to f on the interval $[z, x]$ to conclude that there is a number u with $z < u < x$ for which

$$f(x) - f(z) \ = \ f'(u)(x - z) \quad .$$

Since $f(z) = 0$ and $f'(u) > 0$, one obtains

$$x - z \ = \ \frac{f(x)}{f'(u)} \quad .$$

Since f' is increasing, we find $f'(u) < f'(x)$, and, therefore, $f(x)/f'(x) < f(x)/f'(u)$. Consequently, $z < x' < x$.

In view of (2) one has

$$z < \ldots < x_n < \ldots < x_2 < x_1 \quad .$$

Letting

$$x_* \ = \ \inf_{(n \geq 1)} \ \{x_n\} \ ,$$

one has

$$x_* = \lim_{n\to\infty} x_n ,$$

and, therefore, taking the limit as $n \to \infty$ on both sides of the relation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} ,$$

one finds that $f(x_*) = 0$. Since by (1) there is only one root of f in I, it follows that $x_* = z$.

In the proof of (2) we saw that for x in I on the right side of z the distance from x to z is $f(x)/f'(u)$, where $z < u < x$. Since c is on the concave side of the graph of f, i.e., $c < z$, we find also $c < u$, hence, $f'(c) < f'(u)$. Consequently,

$$\text{the error} = x - z = \frac{f(x)}{f'(u)} \leq \frac{f(x)}{f'(c)} \quad .$$