The Mathematics of Boris Korenblum
Past, Present, and Future

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Highlights

Main areas of research:

- Singular integrals and Fourier series
- Tauberian theorems
- Quasi-analytic functions
- Zero sets and factorization
- Cyclicity in function classes
- Bergman spaces
- Tomography

Samples of significant achievements:

- UAlbany President's Award for Excellence in Research.
- Two Acta papers.
- Invited Address at the International Congress of Mathematicians, Helsinki, 1978.

Singular integrals

In modern harmonic and real analysis, the term “singlar integral” means something like the Hilbert transform:

$$Hf(x) = (p.v.) \int_{-\infty}^{\infty} \frac{f(t)\,dt}{x-t},$$

and more generally, integral operators defined by

$$S_K f(x) = \int_{-\infty}^{\infty} K(x-t) f(t)\,dt,$$

where $K(x)$ is a kernel function satisfying certain growth and symmetry conditions.
However, in the late 1940’s, the term “singular integral” meant something different: integral transforms with delta-type kernels such as the Poisson kernel and the Fejer kernel. Boris’s contributions in this area concerned this type of kernel functions.

Singular integrals

Korenblum's main publications about singular integrals:

- On the representation of functions of L^p classes by singular integrals at Lebesgue points, *Dokl. Akad. Nauk SSSR* **58** (1947), 973-976.
- On the convergence of singular integrals for some general classes of summable functions, *Sbornik Trudov Mat. Inst. Akad. Nauk. Ukr. SSR* **11** (1948), 60-82
- On certain non-linear problems in the theory of singular integrals, *Dokl. Akad. Nauk SSSR* **62** (1948), 17-20.
- A note on Calderon-Zygmund singular integral convolution operators, *Bull. Amer. Math. Soc.* **16** (1987), 271-273, with J. Bruna.

The above papers contain necessary and sufficient conditions for a sequence $\{\varphi_n\}$ of kernel functions to satisfy

$$\lim_{n\to\infty}\int_0^1 \varphi_n(x)f(x)\,dx = A$$

whenever $f \in L^p$ satisfies

$$\lim_{h\to 0}\frac{1}{h}\int_0^h |f(x)-A|^p\,dx = 0,$$

where $p \geq 1$ is any fixed exponent.
The case $p = 1$ was resolved by D. K. Faddeyev in 1936. Some of the results in the papers above were rediscovered much later by Karoly Tandori.

Tauberian Theorems

There is a classical result in calculus/complex analysis (called Abel's theorem) which states that if $\sum a_n x^n$ has radius of convergence equal to 1 and if $\sum a_n$ converges, then

$$\lim_{x\to 1^-} \sum a_n x^n = \sum a_n.$$

Partial inverses to such a result, and later much more general results of this type, are called Tauberian theorems.

Alfred Tauber's original result states that if $a_n = o(1/n)$ and if

$$\lim_{x\to 1^-} \sum a_n x^n = L,$$

then $\sum a_n = L$. Littlewood later relaxed the condition to $a_n = O(1/n)$.

The most important Tauberian theorem was due to N. Wiener, stated in terms of Lebesgue integrable functions on $(0, \infty)$, which was the main motivation for Gelfand's theory of normed rings (Banach algebras).

Tauberian Theorems

Korenblum's main publications on Tauberian theorems:

- On theorems of Tauberian type, *Dokl. Akad. Nauk SSSR* **64** (1949), 449-452.
- Theorems of Tauberian type for a class of Dirichlet series, *Dokl. Akad. Nauk SSSR* **81** (1951), 725-727.
- A general Tauberian theorem for the ratio of functions, *Dokl. Akad. Nauk SSSR* **88** (1953), 745-748.
- A general Tauberian theorem for the ratio of functions, *Uspekhi Mat. Nauk* **8** (1953), 151-153.

Tauberian Theorems

Korenblum's main publications on Tauberian theorems:

- Generalization of Wiener's Tauberian theorem and the spectrum of fast growing functions, *Dokl. Akad. Nauk SSSR* **111** (1956), 280-282.
- A generalization of Wiener's Tauberian theorem and harmonic analysis of rapidly growing functions, *Trudy Moskov. Mat. Obshch* **7** (1958), 121-148.
- A generalization of Wiener's Tauberian theorem and spectrum of rapidly growing functions, Proceedings of the 3rd All-Union Mathematical Congress, Moscow, 1959 (Volume 4, pages 56-58).
- An application of Tauberian theorems to Toeplitz operators, *J. Operator Theory* **33** (1995), 353-361, with K. Zhu.

Tauberian Theorems

Tauberian theorems in the last paper above:

Theorem

Suppose $f \in L^\infty[0,1]$ *and* $a_n = (n+1)\int_0^1 f(r)r^n\,dr$*. Then* $a_n \to 0$ *iff*

$$\lim_{t\to 1}(1-t)^2\sum_{n=0}^\infty (n+1)a_n t^n = 0.$$

Theorem

Suppose $f \in L^\infty[0,1]$ *and* $a_n = (n+1)\int_0^1 f(r)r^n\,dr$*. Then* $a_n \to 0$ *iff*

$$\lim_{x\to 1}\frac{1}{1-x}\int_x^1 f(t)\,dt = 0.$$

Boris's results on ratio tauberian theorems (among his best theorems!) have been well cited and used in differential equations, distribution of zeros, and probability.

Quasi-analyticity

Let f be analytic on $[a, b]$. The classical identity theorem states that if $f = 0$ on some $(c, d) \subset [a, b]$, then $f = 0$ on $[a, b]$. Note that the assumption can also be replaced by $f^{(n)}(c) = 0$ for some fixed $c \in [a, b]$ and all $n \geq 0$.

It turns out that the above identity theorem (both versions) holds for certain classes of functions on $[a, b]$ not necessarily analytic. Such classes are called quasi-analytic classes.

To formulate a more precise notion of quasi-analyticity, first observe that an infinitely differentiable function f on $[a, b]$ is analytic there if and only if there exists a constant $K = K(f)$ such that $|f^{(n)}(x)| \leq K^n n!$ for all $n \geq 0$ and $x \in [a, b]$.

Quasi-analyticity

For any sequence $\{M_n\}$ of positive numbers let $C\{M_n\}$ denote the class of infinitely differentiable functions f on $[a, b]$ with the property that there is a constant $K = K(f) > 0$ such that $|f^{(n)}(x)| \leq K^n M_n$ for all $n \geq 0$ and $x \in [a, b]$. In particular, $C\{n!\}$ is the class of analytic functions on $[a, b]$.

Hadamard's Problem (1912): Characterize those sequences $\{M_n\}$ for which $C\{M_n\}$ is quasi-analytic (i.e., the identity theorem holds for it).

Example (Denjoy 1921): $M_n = n!(\ln n)^n$, $M_n = n!(\ln n)^n(\ln \ln n)^n, \cdots$, all give rise to quasi-analytic classes.

Necessary and sufficient conditions were later obtained by T. Carleman. Note that the notion of quasi-analyticity also makes sense on planar domains.

Quasi-analyticity

Korenblum's publications about quasi-analyticity:

- Phragmen-Lindelöf type theorems for quasi-analytic classes of functions, Issledovaniya po sovrememnym probleman teorii funkciikompleksnogo peremennogo, Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960, 526-531.
- Quasi-analytic classes of functions in a circle, *Dokl. Akad. Nauk SSSR* **164** (1965), 36-39.

The second paper above has been widely cited. It gives necessary and sufficient conditions for a class $|f^{(n)}(z)| < A_n$, $n = 0, 1, 2, \cdots$, to be quasi-analytic in a disk.

Although Boris were making significant contributions to complex and harmonic analysis before my generalization of analysts were even born, most modern analysts get to know (of) Boris through his two Acta papers:

- An extension of the Nevanlinna theory, *Acta Math.* **135** (1975), 187-219.
- A Beurling type theorem, *Acta Math.* **138** (1977), 265-293.

In particular, Boris was invited to present his work in this area at the International Congress of Mathematicians in Helsinki, 1978:

- Analytic functions of unbounded characteristic and Beurling algebras, *Proc. Intl. Congress Mathematicians*, Acad. Sci. Fenn., 653-658.

The Acta Papers

A good number of analysts know of Boris's two Acta papers, but very few have managed to fully understand them. The following is quoted from Kristian Seip's openning speech at the "Korenblum Fiesta" (Nov 2003, Barcelona, Spain), in celebration of Korenblum's 80th birthday.

"There are probably not many people who have really penetrated all aspects of those papers. I have personally been very much inspired by those papers, which contain amazing and deep ideas. One of the most striking aspects is the way linear programming enters the study of zero sets for analytic functions. As far as I know, there is no other way of getting such precise estimates for zero sequences of Bergman spaces."

The Acta Papers

By the way, Kristian Seip is a former President of the Norwagean Mathematical Society, one of the current editors of Acta, and one of the few who may have grasped most aspects of those papers. In particular, using ideas and concepts developed in Korenblum's Acta papers, Seip was able to obtain a complete characterization of interpolating and sampling sequences for several classes of Bergman type spaces. This work earned him a spot at the International Congress of Mathematicians in Zurich, 1994.

Another side remark: Walter Hayman (a distinguished complex analyst at Queen's College, London, and one of Korenblum's long time collaborators) ended his review of the first Acta paper in Math. Reviews this way, "The above sketch must suffice to give some idea of this extremely complicated and profound paper".

The Acta Papers

To get a rough idea of what kind of mathematics Korenblum's Acta papers were about, consider the following example and analogue.

Let N denote the Nevanlinna class consisting of analytic functions in the unit disk satisfying the condition

$$\sup_{0<r<1} \int_0^{2\pi} \log^+ |f(re^{it})|\, dt < \infty.$$

These are called functions of bounded characteristics. The following result is classical.

Theorem

- *A sequence $\{z_n\}$ is the zero set of a function in N iff it satisfies the Blaschke condition: $\sum(1 - |z_n|) < \infty$.*
- *Every function $f \in N$ admits a unique factorization: $f = \varphi F$, where φ is a so-called inner function and F is a so-called outer function.*

A simplistic way of summarizing Korenblum's two Acta papers is this: The two papers extended the above theorem to the class $A^{-\infty}$ consisting of analytic functions f in the unit disk with the following property: there exists a positive number c such that

$$\sup_{|z|<1} (1-|z|)^c |f(z)| < \infty.$$

Such functions generally have unbounded characteristics.

A complete description of zero sequences for $A^{-\infty}$ was obtained in terms of a certain geometric density. This notion of density was later used by Seip to characterize interpolating and sampling sequences for Bergman type spaces.

Boris also extended the notions of inner and outer functions to the $A^{-\infty}$ setting. This was based on his new concept of κ-singular measures, "measures" that are concentrated on very thin sets on the unit circle.

For $0 < p < \infty$ the Bergman space A^p consists of analytic functions in the unit disk $\mathbb{D}$ that belong to $L^p(\mathbb{D}, dA)$, where dA is area measure on $\mathbb{D}$. Thus

$$A^p = L^p(\mathbb{D}, dA) \cap H(\mathbb{D}),$$

where $H(\mathbb{D})$ is the space of all analytic functions in $\mathbb{D}$.
The definition of Bergman spaces is extremely simple, but a lot of fundamental problems remain unsolved for them. Here are three examples (open problems):

- How to characterize the zero sets of Bergman spaces?
- How to characterize the cyclic vectors of Bergman spaces?
- How to characterize the invariant subspaces of Bergman spaces?

Bergman Spaces-Zero Sets

Let X be any class of analytic functions in $\mathbb{D}$. A sequence $\{z_n\}$ in $\mathbb{D}$ is called a zero set for X if there exists a function $f \in X$, not identically zero, such that $\{z_n\}$ is the zero sequence of f, counting multiplicity.

Classical example: Let H^∞ denote the space of bounded analytic functions in $\mathbb{D}$. Then a sequence $\{z_n\}$ in $\mathbb{D}$ is a zero set for H^∞ if and only if it satisfies the Blaschke condition, that is, $\sum(1-|z_n|) < \infty$. The same result holds for other Hardy spaces H^p as well.

It is still an open problem to characterize the zero sets for A^p. The best partial results so far are still corollaries that can be derived from Korenblum's Acta papers, despite the effort by numerous analysts over several decades. Boris had a necessary condition and a sufficient condition that are ϵ-apart. Nobody knows how to close this small gap.

Bergman Spaces-Cyclic Vectors

Let X be a topological space of analytic functions in $\mathbb{D}$ with the property that the function $zf(z)$ is in X whenever $f \in X$.

Obviously, $pf \in X$ whenever $f \in X$ and p is a polynomial.

A function $F \in X$ is called a cyclic vector if the the polynomial multiples of f are dense in X. In particular, if the polynomials are dense in X, then the constant function 1 is a cyclic vector.

Classical example: the cyclic vectors of the Hardy spaces H^p are exactly the so-called outer functions (defined in terms measures on the unit circle that are absolutely continuous with respect to Lebesgue measure).

It is still an open problem to characterize the cyclic vectors in Bergman spaces. Again, the best results so far in this area are corrolaries that can be derived from Korenblum's Acta papers.

Bergman Spaces-Invariant Subspaces

Let X be as above. A closed subspace S of X is called an invariant subspace if the function $zf(z)$ is still in S whenever $f \in S$.

Classical example (Beurling's theorem): A closed subspace I of the Hardy space H^p is an invariant subspace if and only if $I = \varphi H^p$, where φ is an inner function (defined in terms of zero sets and measures on the unit circle that are singular with respect to Lebesgue measure).

Again, it is still an open problem to characterize/classify the invariant subspaces of A^p. The special case $p = 2$ is especially interesting and important, not only in function theory, but also in general operator theory.

Korenblum's insight and personal guidance again played an important role in the developments in this area in recent years.

Bergman Spaces-Brief History

The theory of Bergman spaces consists of two main parts: operator theory and function theory.

The operator theoretic part became “mature” by the mid-1980’s. My book “Operator Theory in Function Spaces” (1990) contains the main achievements in this area of analysis.

Because of Korenblum’s work on $A^{-\infty}$ and possibly my own work about operator theory on Bergman spaces, Haakan Hedenmalm (a young analyst then from Sweden interested in Bergman spaces) visited our department for a semester around 1990. Right after the visit, influenced by conversations and exchanges of ideas with Boris, Hedenmalm proved a ground breaking result: certain extremal functions G constructed from zero sets of the Bergman space A^2 have the property that $\|Gf\|_2 \geq \|f\|_2$ for $f \in H^\infty$ (which is dense in A^2).

Hedenmalm has acknowledged Boris’s influence on several different occasions.

Hedenmalm's result ignited a firestorm in the development of function theory of Bergman spaces.

- Contractive zero divisors: Hedenmalm, Duren, Khavinson, Shapiro, Sundberg.
- Cyclic vectors: Hedenmalm, Borichev.
- Inner-outer factorization: Korenblum et el.
- Invariant subspaces: Aleman, Richter, Shimorin, Sundberg.
- Zero sets, interpolating sets, sampling sets: Seip.

Two books appeared to summarize these developments:

- Hedenmalm, Korenblum, and Zhu, *Theory of Bergman Spaces*, Springer, New York, 2000.
- Duren and Schuster, *Bergman Spaces*, AMS, 2004.

Bergman Spaces-Recent Publications

Korenblum's publications about Bergman spaces in recent years:

- Unimodular Möbius invariant contractive divisors for the Bergman space, The Gohberg Anniversary Collection II, *Operator Theory Adv. Appl.* **41**, Birkhäuser, Basel, 1989, 353-358.
- Transformation of zero sets by contractive operators in the Bergman space, *Bull. Sci. Math.* **114** (1990), 385-394.
- A maximum principle for the Bergman space, *Publ. Mat.* **35** (1991), 479-486.
- On Toeplitz-invariant subspaces of the Bergman space, *J. Funct. Anal.* **111** (1993), 76-96, with M. Stessin.
- Outer functions and cyclic elements in Bergman spaces, *J. Funct. Anal.* **115** (1993), 104-118.

Bergman Spaces-Recent Publications

- Majorization and domination in Bergman spaces, *Proc. Amer. Math. Soc.* **117** (1993), 153-158, with K. Richards.
- Totally monotone functions with applications to the Bergman space, *Trans. Amer. Math. Soc.* **337** (1993), 795-806, with R. O'Neil, K. Richards, and K. Zhu.
- Beurling type invariant subspaces of the Bergman space, *J. London Math. Soc.* **53** (1996), 601-614, with H. Hedenmalm and K. Zhu.
- Projective generators in Hardy and Bergman spaces, *Bull. Sci. Math.* **124** (2000), 453-445, with T. Lance and M. Stessin.
- *Theory of Bergman Spaces*, Springer, New York, 2000, with H. Hedenmalm and K. Zhu.
- Blaschke sets for Bergman spaces, *Contemp. Math.* **404** (2006), 145-152.

Starting from about two years ago, Boris has been working with Alexander Aleman on several new projects, including the notions of derivation-invariant and Volterra-invariant subspaces of several function spaces, resulting in the following papers (and possibly more to come):

- Derivation-invariant subspaces of C^∞, *CMFT* **8** (2008), 493-512, with A. Aleman.
- Volterra-invariant subspaces of H^p, *Bull. Sci. Math.* **132** (2008), 510-528, with A. Aleman.

Nobody really knows what the future holds for us, but I am sure that several problems and ideas closely associated with Boris will continue to be pursued by future generations of complex analysts. These include

- Zero sets for Bergman spaces.
- Cyclic vectors for Bergman spaces.
- Maximum principle for Bergman spaces.
- The notion of domination in Bergman spaces.
- The notion of inner and outer functions in Bergman spaces.
- Invariant subspaces of Bergman spaces.
- The $A^{-\infty}$ theory.

The Other Side of Boris

We all know that Boris is an accomplished mathematician, but very few of us know that Boris has also made high profile contributions to other areas of science. I mention two such contributions here:

- Computed Tomography.
- Physics.

By the way, before 1974, Boris was known as B.I. Korenblyum, with an extra y.

Physics

As has been common for mathematicians from the former Soviet union, Boris was well educated in applied areas of mathematics and has done significant research in these areas. I do not have a complete list of Boris's publications outside mathematics, but here I list two items, one from a long time ago and one from just a few years ago:

- On the mathematical theory of the optimal amplitude-phase modulation method, *Sbornik Trudov Inst. Elektrotekhn, Akad. Nauk. Ukr. SSR* **7** (1951), 96-104, with M.G. Krein and S.I. Telelbaum.
- Classical properties of low-dimensional conductors: giant capacitance and non-ohmic potential drop, *Physics Review Letters*, 2002, with Emmanuel Rashba.

Computed Tomography

CT scanners are now very common in medical practices. G.N. Hounsfield received the 1979 Nobel prize in Physiology and Medicine ”for his contruction of a machine used to X-ray computed tomography in a clinical environment”.

To highlight Korenblum’s contributions in this area, I quote the following passage from the article ”In the beginning” in the book *The Physics of Medical Imaging* edited by Steve Webb (1988): “It is perhaps less well known that a CT scanner was built in Russia in 1958. Korenblyum et al (1958) published the mathematics of reconstruction from projections together with experimental details and wrote: ‘At the present time at Kiev Polytechnic Institute, we are constructing the first experimental apparatus for getting X-ray images of thin sections by the scheme described in this article’. This was an analogue reconstruction method, based on a television detector and a fan-beam source of x-ray. Earlier reports from Russia have also been found (e.g. Tele’Baum 1957).”

Computed Tomography

The following is another quote from a short memo "Historical Note on Computed Tomography" written by H. Barret, W. Hawkins, and M. Joy:

"In 1958, Korenblyum, Telel'baum, and Tyutin gave a detailed account of the theory, including a method of fan-beam correction, equivalent to reordering the data, and a discussion of a practical way of handling the singularity in the integrand of the inverse Radon transform. They also presented a block diagram of a television-based analog computing system for implementing the reconstruction....

One of us (W.G.H.) programmed the reconstruction algorithm of Korenblyum et al and found that it indeed performs satisfactorily.... An English translation of the following article is available on request from H.H. Barret."

Korenblyum BI, Tele'baum SI, Tyutin AA, About one scheme of tomography, *Bull. Inst. Higher Education–Radiophysics (Izvestiya Vyshikh Uchebnykh Zavedenii-Radiofizika)* 1958;

Some Statistics

- Publications (unofficial): 85++
- Ph.D. Students (official): 10 (V.S. Korolevich, V.A. Dubovik, L.V. Shamraeva, I.P. Pobyvanets, K. Samotij, G. Bomash, A. Mascuilli, J. Panariello, J. Racquet, C. Beneteau)
- Collaborators (AMS count): 29
- Citations (AMS count): 296
- Countries worked/lived in: 3 or 4
- Languages spoken: 3 or 4

Thank You

HAPPY RETIREMENT, BORIS!